On partial regularity for the $3D$ non-stationary Hall magnetohydrodynamics equations on the plane

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Abstract

We study partial regularity of weak solutions of the 3D valued non-stationary Hall magnetohydrodynamics equations on $\mathbb{R}^2$. In particular we prove the existence of a weak solution whose set of possible singularities has the space-time Hausdorff dimension at most two.

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1 Introduction and the main theorem

We consider the homogeneous incompressible Hall magnetohydrodynamics(Hall-MHD) equations:

$$\begin{cases} \dfrac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{u} \cdot \nabla \boldsymbol{u} + \nabla p = (\nabla \times \boldsymbol{B}) \times \boldsymbol{B} + \nu \Delta \boldsymbol{u} + \boldsymbol{f}, \\ \dfrac{\partial \boldsymbol{B}}{\partial t} - \nabla \times (\boldsymbol{u} \times \boldsymbol{B}) + \nabla \times ((\nabla \times \boldsymbol{B}) \times \boldsymbol{B}) = \mu \Delta B + \nabla \times \boldsymbol{g}, \\ \nabla \cdot \boldsymbol{u} = 0, \ \nabla \cdot \boldsymbol{B} = 0, \end{cases}$$

where the three dimensional vector fields $\boldsymbol{u} = \boldsymbol{u}(x,t)$ and $\boldsymbol{B} = \boldsymbol{B}(x,t)$ are the fluid velocity and the magnetic field respectively. The scalar field $p = p(x,t)$ is the pressure, while the positive constants ν and μ represent the viscosity and the magnetic resistivity

respectively. The given vector fields $\boldsymbol{f}$ and $\nabla \times \boldsymbol{g}$ are external forces on the magnetically charged fluid flows. The system has been studied first by Lighthill [13] in 1960. We notice that comparing with the usual MHD system, the Hall-MHD system contains the extra term $\nabla \times ((\nabla \times \boldsymbol{B}) \times \boldsymbol{B})$, called the Hall term. The inclusion of this term is essential in understanding the phenomena of magnetic reconnection, meaning the change of the topology of magnetic field lines. This is observed in real physical situations such as space plasma [9, 11], star formation [21] and neutron star [19]. There are also many other physical phenomena that requires the Hall-MHD system to describe them (see e.g. [15, 20, 16] and the references therein). The Hall term is quadratically nonlinear, containing the second order derivative, and it causes major difficulties in the mathematical study of the Hall-MHD system. Thanks to the orthogonality in L^2 of the Hall term with $\boldsymbol{B}$, however, the energy inequality similar to the usual MHD case holds true. Using this fact the construction of the global in time weak solution can be achieved without any difficulties, as has been shown in [1]. Observing similar cancellation properties of the Hall term, the local in time well-posedness as well as the global in time well-posedness for small initial data was also established in [3], and has been refined in [4]. Regarding the question of energy conservation for weak solutions in the inviscid case we refer to [7]. For a special form of axially symmetric initial data the authors of [8] proved the global in time existence of classical solutions to the system. On the other hand, the optimal temporal decay estimates are obtained in [5].

Concerning the regularity of weak solutions, one can expect that the problem is more difficult than the Navier-Stokes equations and the usual MHD system. Even the problem of regularity of stationary weak solution has essential difficulty with current methods of the regularity theory, which is contrary to the case of the stationary Navier-Stokes equations. The partial regularity of stationary weak solutions has been obtained recently by the current authors (cf. [6]). In the present paper we investigate the partial regularity of weak solutions of *the non-stationary system*. For the Navier-Stokes equations there are many publications on this direction of study (see e.g. [18, 2, 12, 14, 23]). In the case of the 3D Hall-MHD system in $\mathbb{R}^3$, however, we encounter again essential difficulties in constructing suitable weak solutions, satisfying desired form of localized energy inequality.

In the current paper we focus on the case of 3D valued Hall-MHD system on the plane, which is sometimes called the $2\frac{1}{2}$ dimensional system. Physically the situation corresponds to the full 3D system having the translational symmetry in the x_3 direction. In this case, as will be shown in detail below, although we cannot construct suitable weak solution, satisfying the localized energy inequality, instead, we could construct an approximate system, for which we can deduce Caccioppoli-type inequalities to obtain "approximate singular set", and then by passing to a limit in an appropriate sense, we can show that there exists a possible singular set for the limiting weak solution, whose Hausdorff dimension is at most two. When we try to apply the similar idea to the full 3D non-stationary system defined on $\mathbb{R}^3$, we have difficulty in constructing a sequence of the approximate weak solutions, the compactness of which is strong enough to pass to the limit. Therefore, we leave the proof of partial regularity of the full 3D non-stationary system as an open problem. We now formulate our problem more precisely, and state our main result.

We concentrate on the following $3D$ valued Hall-MHD system in $Q = \mathbb{R}^2 \times (0,T)$.

$$\partial_t \boldsymbol{u} + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u} - \Delta\boldsymbol{u} = -\nabla p + (\nabla\times\boldsymbol{B})\times\boldsymbol{B} + \boldsymbol{f}, \tag{1.1}$$

$$\partial_t \boldsymbol{B} + \nabla\times(\boldsymbol{B}\times\boldsymbol{u}) - \Delta\boldsymbol{B} = -\nabla\times((\nabla\times\boldsymbol{B})\times\boldsymbol{B}) + \nabla\times\boldsymbol{g}, \tag{1.2}$$

$$\nabla\cdot\boldsymbol{u} = 0, \quad \nabla\cdot\boldsymbol{B} = 0 \tag{1.3}$$

together with the initial condition

$$\boldsymbol{u} = \boldsymbol{u}_0, \quad \boldsymbol{B} = \boldsymbol{B}_0 \quad \text{on} \quad \mathbb{R}^2\times\{0\}, \tag{1.4}$$

which satisfy

$$\nabla\cdot\boldsymbol{u}_0 = \nabla\cdot\boldsymbol{B}_0 = 0 \quad \text{on} \quad \mathbb{R}^2. \tag{1.5}$$

Here, $\boldsymbol{u} = (u_1,u_2,u_3), \boldsymbol{B} = (B_1,B_2,B_3)$, where $u_j = u_j(x_1,x_2,t), B_j = B_j(x_1,x_2,t), j = 1,2,3$, and $p = p(x_1,x_2,t)$, $(x,t) = (x_1,x_2,t) \in Q$. Note that we set $\nu = \mu = 1$ for convenience. For the definition of weak solution see Definition 1.1 below. The aim of the present paper is to prove the existence of a weak solution to the Hall-MHD system (1.1)–(1.3), which is Hölder continuous outside of a possible singular set together with the estimation of its Hausdorff dimension. We set $L^2_{\mathrm{div}} = \{\boldsymbol{u}\in L^2 \,|\, \nabla\cdot\boldsymbol{u} = 0\}$, where the derivative is defined in the sense of distribution. We also define $V^2(Q) = L^\infty(0,T;L^2)\cap L^2(0,T;W^{1,2})$. By $V^2_{\mathrm{div}}(Q)$ we denote the space of all $\boldsymbol{u}\in V^2(Q)$ such that $\nabla\cdot\boldsymbol{u} = 0$ in the sense of distribution in Q.

Notice that using the formula $(\boldsymbol{u}\cdot\nabla)\boldsymbol{u} = (\nabla\times\boldsymbol{u})\times\boldsymbol{u} + \frac{1}{2}|\boldsymbol{u}|^2$, one can rewrite (1.1) into

$$\partial_t\boldsymbol{u} + (\nabla\times\boldsymbol{u})\times\boldsymbol{u} - \Delta\boldsymbol{u} = -\nabla\Big(p + \frac{|\boldsymbol{u}|^2}{2}\Big) + (\nabla\times\boldsymbol{B})\times\boldsymbol{B} + \boldsymbol{f} \quad \text{in} \quad Q. \tag{1.6}$$

Applying $\nabla\times$ to the both sides of the above, we get

$$\partial_t\boldsymbol{\omega} + \nabla\times(\boldsymbol{\omega}\times\boldsymbol{u}) - \Delta\boldsymbol{\omega} = \nabla\times((\nabla\times\boldsymbol{B})\times\boldsymbol{B}) + \nabla\times\boldsymbol{f} \quad \text{in} \quad Q, \tag{1.7}$$

where $\boldsymbol{\omega}$ stands for the vorticity $\nabla\times\boldsymbol{u}$. Taking the sum of (1.2) and (1.7), we are led to

$$\partial_t\boldsymbol{V} + \nabla\times(\boldsymbol{V}\times\boldsymbol{u}) - \Delta\boldsymbol{V} = \nabla\times(\boldsymbol{f}+\boldsymbol{g}) \quad \text{in} \quad Q, \tag{1.8}$$

where

$$\boldsymbol{V} = \boldsymbol{B} + \boldsymbol{\omega}. \tag{1.9}$$

Since $\nabla\cdot\boldsymbol{V} = 0$, there exists a solenoidal potential $\boldsymbol{v}$ such that $\nabla\times\boldsymbol{v} = \boldsymbol{V}$. From (1.8) we deduce that $\boldsymbol{v}$ solves the following system in Q,

$$\nabla\cdot\boldsymbol{v} = 0, \tag{1.10}$$

$$\partial_t\boldsymbol{v} + (\boldsymbol{v}\cdot\nabla)\boldsymbol{v} - \Delta\boldsymbol{v} = -\nabla\pi + (\nabla\times\boldsymbol{v})\times\boldsymbol{b} + \boldsymbol{f} + \boldsymbol{g}, \tag{1.11}$$

where $\boldsymbol{b} = \boldsymbol{v} - \boldsymbol{u}$. Clearly, $\nabla\times\boldsymbol{b} = \boldsymbol{B}$.

We now introduce the notion of a weak solution to the system (1.1)–(1.5).

Definition 1.1. Let $\boldsymbol{f}, \boldsymbol{g} \in L^2(Q)$. We say $(\boldsymbol{u}, p, \boldsymbol{B}) \in V^2_{\mathrm{div}}(Q) \times L^2(0,T; L^2_{\mathrm{loc}}) \times V^2_{\mathrm{div}}(Q)$ is a *weak solution* to (1.3)–(1.4) if

$$\begin{aligned}
&\int_Q (-\boldsymbol{u} \cdot \partial_t \boldsymbol{\varphi} + \nabla \boldsymbol{u} : \nabla \boldsymbol{\varphi} - \boldsymbol{u} \otimes \boldsymbol{u} : \nabla \boldsymbol{\varphi}) dx dt \\
&\quad = \int_Q p \nabla \cdot \boldsymbol{\varphi} dx dt + \int_Q ((\nabla \times \boldsymbol{B}) \times \boldsymbol{B} + \boldsymbol{f}) \cdot \boldsymbol{\varphi} dx dt + \int_{\mathbb{R}^2} \boldsymbol{u}_0 \cdot \boldsymbol{\varphi}(0) dx,
\end{aligned} \tag{1.12}$$

$$\begin{aligned}
&\int_Q (\boldsymbol{B} \cdot \partial_t \boldsymbol{\varphi} + \nabla \boldsymbol{B} : \nabla \boldsymbol{\varphi} + \boldsymbol{B} \times \boldsymbol{u} : \nabla \times \boldsymbol{\varphi}) dx dt \\
&\quad = \int_Q ((\nabla \times \boldsymbol{B}) \times \boldsymbol{B} + \boldsymbol{g}) \cdot \nabla \times \boldsymbol{\varphi} dx dt + \int_{\mathbb{R}^2} \boldsymbol{B}_0 \cdot \boldsymbol{\varphi}(0) dx
\end{aligned} \tag{1.13}$$

for all $\boldsymbol{\varphi} \in C^\infty_{\mathrm{c}}(\mathbb{R}^2 \times [0,T))$. Here we used the notation $\boldsymbol{A} : \boldsymbol{B} = \sum_{i,j=1}^3 A_{ij} B_{ij}$ for matrices $\boldsymbol{A}, \boldsymbol{B} \in \mathbb{R}^{3\times 3}$.

By $\mathcal{M}^{2,\lambda}_{\mathrm{loc}}(Q)$ we denote the local Morrey space, which is defined in Section 3 below. Our main result is the following theorem.

Theorem 1.2. *Let $\boldsymbol{u}_0 \in L^2_{\mathrm{div}}, \boldsymbol{B}_0 \in L^2$ and $\boldsymbol{f}, \boldsymbol{g} \in L^2(Q)$. Moreover, we suppose that $\boldsymbol{g} \in \mathcal{M}^{2,\lambda}_{\mathrm{loc}}(Q)$ for some $2 < \lambda < 4$. Then, there exists a weak solution $(\boldsymbol{u}, p, \boldsymbol{B}) \in V^2_{\mathrm{div}}(Q) \times L^2(0,T; L^2_{\mathrm{loc}}) \times V^2_{\mathrm{div}}(Q)$ of* (1.1)–(1.5) *being α-Hölder continuous outside of a closed subset set $\Sigma(\boldsymbol{B}) \subset Q$ of Hausdorff dimension less than or equal to two, where $0 < \alpha < \frac{\lambda - 2}{2}$.*

The paper is organized as follows. In Section 2 we discuss local estimates of weak solutions to the approximate system related to (1.2) involving the magnetic field $\boldsymbol{B}$. Thanks to the validity of the local energy equality (see (2.4) below) we are able to establish a Caccioppoli-type inequality, which plays a central role in the proof of the fundamental estimate in Section 3 (cf. Lemma 3.2). To achieve this result we make use of an indirect argument together with the fundamental estimate which holds true for the corresponding linear limit system (cf. Lemma 3.1). The aim of section Section 4 is the construction of an approximate solution to system (1.1)–(1.5) along with the required *a priori* estimates. Furthermore, passing to the limit in the approximate system we get a weak solution to (1.1)–(1.5). In Section 5 we prove that the weak solution constructed in Section 4 fulfills the required partial regularity property stated in Theorem 1.2, the main result of the paper. We wish to remark that even for the weak solution to the system under consideration constructed in a suitable way, a corresponding local energy inequality similar to the case of the Navier-Stokes equation may not be available. For this reason in the proof of the main theorem we are only able to work on the approximate solutions using Lemma 3.2. The estimation of the parabolic Hausdorff dimension of the singular set is obtained by Theorem 5.1, the proof of which can be found at the end of Section 5. For readers convenience we added an appendix which contains the definition of the parabolic Hölder space $C^{\alpha,\alpha/2}(Q)$, the parabolic version of the Poincaré inequality and an algebraic lemma which will be used in the proof of Theorem 5.1.

2 Caccioppoli-type inequality for the approximate $\boldsymbol{B}$ system

Let $\boldsymbol{g}, \boldsymbol{u} \in L^2(Q)$ be given. For fixed $0 < \delta < 1$ we consider the following system for $\boldsymbol{B}$ approximating (1.2)

$$
\begin{aligned}
&\partial_t \boldsymbol{B} - \Delta \boldsymbol{B} \\
&= -\nabla \times \Big(\nabla \times \boldsymbol{B} \times \frac{\boldsymbol{B}}{1+\delta|\boldsymbol{B}|} \Big) + \nabla \times \Big(\boldsymbol{u} \times \frac{\boldsymbol{B}}{1+\delta|\boldsymbol{B}|} \Big) + \nabla \times \boldsymbol{g} \quad \text{in} \quad Q.
\end{aligned}
\tag{2.1}
$$

We start our discussion with the following notions of a weak solution to (2.1).

Definition 2.1. A vector field $\boldsymbol{B} \in V^2(Q)$ is said to be a *weak solution to* (2.1) if

$$
\begin{aligned}
&\int_Q (-\boldsymbol{B} \cdot \partial_t \boldsymbol{\varphi} + \nabla \boldsymbol{B} : \nabla \boldsymbol{\varphi}) dxdt \\
&= -\int_Q (\nabla \times \boldsymbol{B} - \boldsymbol{u}) \times \frac{\boldsymbol{B}}{1+\delta|\boldsymbol{B}|} \cdot \nabla \times \boldsymbol{\varphi} dxdt + \int_Q \boldsymbol{g} \cdot \nabla \times \boldsymbol{\varphi} dxdt
\end{aligned}
\tag{2.2}
$$

for all $\boldsymbol{\varphi} \in C^\infty_{\mathrm{c}}(\Omega)$.

Remark 2.2. Let $\boldsymbol{B}$ be a weak solution to (2.1). Then, (2.2) yields the existence of the distributional time derivative $\boldsymbol{B}' \in L^2(0,T; W^{-1,2})$, determined by the identity

$$
\begin{aligned}
&\int_{\mathbb{R}^2} \langle \boldsymbol{B}'(s), \boldsymbol{\psi} \rangle dx + \int_{\mathbb{R}^2} \nabla \boldsymbol{B}(s) : \nabla \boldsymbol{\psi} dx \\
&= -\int_{\mathbb{R}^2} (\nabla \times \boldsymbol{B}(s) - \boldsymbol{u}(s)) \times \frac{\boldsymbol{B}(s)}{1+\delta|\boldsymbol{B}(s)|} \cdot \nabla \times \boldsymbol{\psi} dx + \int_{\mathbb{R}^2} \boldsymbol{g}(s) \cdot \nabla \times \boldsymbol{\psi} dx
\end{aligned}
\tag{2.3}
$$

for all $\boldsymbol{\psi} \in W^{1,2}(\mathbb{R}^2)$ and for a.e. $s \in (0,T)$. Inserting $\boldsymbol{\psi}(x,s) = \phi(x,s)(\boldsymbol{B}(x,s) - \boldsymbol{\Lambda})$ into (2.3) with $\phi \in C^\infty_{\mathrm{c}}(Q)$ and a constant vector $\boldsymbol{\Lambda} \in \mathbb{R}^3$, integrating the result over $(0,t)$

($t \in (0,T)$) and using integrating by parts, we obtain the following local energy equality

$$
\begin{aligned}
&\frac{1}{2}\int\limits_{\mathbb{R}^2} \phi(t)|\boldsymbol{B}(t)-\boldsymbol{\Lambda}|^2 dx + \int\limits_0^t \int\limits_Q \phi|\nabla \boldsymbol{B}|^2 dxds \\
&\quad = \frac{1}{2}\int\limits_0^t \int\limits_{\mathbb{R}^2} (\partial_t \phi + \Delta\phi)|\boldsymbol{B}-\boldsymbol{\Lambda}|^2 dxds \\
&\qquad + \int\limits_0^t \int\limits_{\mathbb{R}^2} (\nabla\times\boldsymbol{B}-\boldsymbol{u})\times\frac{\boldsymbol{B}}{1+\delta|\boldsymbol{B}|}\cdot((\boldsymbol{B}-\boldsymbol{\Lambda})\times\nabla\phi) dxds \\
&\qquad + \int\limits_0^t \int\limits_{\mathbb{R}^2} \phi\boldsymbol{u}\times\frac{\boldsymbol{B}}{1+\delta|\boldsymbol{B}|}\cdot\nabla\times\boldsymbol{B} dxds \\
&\qquad + \int\limits_0^t \int\limits_{\mathbb{R}^2} \Big(\phi\boldsymbol{g}\cdot\nabla\times\boldsymbol{B} - \boldsymbol{g}\cdot(\boldsymbol{B}-\boldsymbol{\Lambda})\times\nabla\phi\Big) dxds.
\end{aligned}
\tag{2.4}
$$

First, let us fix some notations which is used throughout the present and subsequent sections. Let $X_0 = (x_0, t_0) \in \mathbb{R}^3$ and $0 < r < +\infty$ by $Q_r = Q_r(X_0)$ we denote the parabolic cylinder $B_r(x_0)\times(t_0 - r^2, t_0)$. Furthermore, for a function $f \in L^1(Q_r)$ we define

$$
f_{r,X_0} := f_{Q_r} = \fint_{Q_r} f dxdt = \frac{1}{\operatorname{mes} Q_r}\int\limits_{Q_r} f dxdt,
$$

where $\operatorname{mes} Q_r$ stands for the three dimensional Lebesgue measure of Q_r.

Let $0 < \rho < r$. We call $\theta \in C^\infty(\mathbb{R}^3)$ a *cut-off function* suitable for Q_r and Q_ρ if $0 \le \theta \le 1$ in $\mathbb{R}^3$, $\theta \equiv 1$ on Q_ρ, $\theta \equiv 0$ in $(\mathbb{R}^3 \setminus B_r)\times(t_0 - r^2, t_0) \cup \mathbb{R}^2 \times (-\infty, t_0 - r^2)$ and $|\partial_t \theta| + |\nabla\theta|^2 + |\nabla^2\theta| \le c(r-\rho)^{-2}$ in $\mathbb{R}^3$.

Now, we state the following Caccioppoli-type inequality.

Lemma 2.3. *Let $\boldsymbol{g} \in L^2(Q), \boldsymbol{u} \in L^4(Q)$ be given, and let $\boldsymbol{B} \in V^2(Q)$ be a weak solution to* (2.1). *Then, for every cylinder $\overline{Q}_r = \overline{Q_r(X_0)} \subset Q$ and $0 < \rho < r$ there holds*

$$
\begin{aligned}
&\operatorname*{ess\,sup}_{t\in(t_0-r^2,t_0)} \int\limits_{B_r} \theta^4|\boldsymbol{B}-\boldsymbol{B}_{r,X_0}|^2 dx + \int\limits_{Q_r} \theta^4|\nabla\boldsymbol{B}|^2 dxdt \\
&\quad \le \frac{c}{(r-\rho)^2}(1+|\boldsymbol{B}_{r,X_0}|^2)\int\limits_{Q_r} |\boldsymbol{B}-\boldsymbol{B}_{r,X_0}|^2 dxdt \\
&\qquad + \frac{c}{r-\rho}\bigg(\int\limits_{Q_r} \theta^{3+\gamma}|\boldsymbol{B}-\boldsymbol{B}_{r,X_0}|^4 dxdt\bigg)^{1/2}\bigg(\int\limits_{Q_r} \theta^{3-\gamma}|\nabla\boldsymbol{B}|^2 dxdt\bigg)^{1/2} \\
&\qquad + c\int\limits_{Q_r} (|\boldsymbol{g}|^2 + \theta^4|\boldsymbol{B}|^2|\boldsymbol{u}|^2) dxdt
\end{aligned}
\tag{2.5}
$$

for all cut-off function θ suitable for Q_r and Q_ρ ($\gamma \in [-3,3]$), and

$$E(\rho)^4 \le \frac{cr^4}{(r-\rho)^4}(1+|\boldsymbol{B}_{r,X_0}|^2)(G(r)^4+F(r)^4)$$
$$+\frac{cr^6}{(r-\rho)^6}(G(r)^6+F(r)^6)$$
$$+\frac{c}{(r-\rho)^2}\left\{\int\limits_{Q_r}|\boldsymbol{g}|^2dxdt+|\boldsymbol{B}_{r,X_0}|^2\int\limits_{Q_r}|\boldsymbol{u}|^2dxdt\right\}(G(r)^2+F(r)^2)$$
$$+\frac{cr^4}{(r-\rho)^4}\int\limits_{Q_r}|\boldsymbol{u}|^4dxdt(G(r)^4+F(r)^4), \tag{2.6}$$

where $c=\mathrm{const}>0$ *denotes a universal constant, and*

$$E(r)=E(r,X_0)=\left(\fint_{Q_r(X_0)}|\boldsymbol{B}-\boldsymbol{B}_{r,X_0}|^4dxdt\right)^{1/4},$$
$$F(r)=F(r,X_0)=\left(r^{-2}\int\limits_{Q_r(X_0)}|\nabla\boldsymbol{B}|^2dxdt\right)^{1/2},$$
$$G(r)=G(r,X_0)=\left(\fint_{Q_r(X_0)}|\boldsymbol{B}-\boldsymbol{B}_{r,X_0}|^2dxdt\right)^{1/2}, 0<r<\sqrt{t_0}.$$

Proof Let $\overline{Q}_r=\overline{Q_r(X_0)}\subset Q$ be a fixed cylinder. For $0<\rho<r$ we take a cut-off function $\theta\in C^\infty(\mathbb{R}^3)$ suitable for Q_r and Q_ρ.

From (2.4) with $\phi=\theta^4$ and $\boldsymbol{\Lambda}=\boldsymbol{B}_{r,X_0}$ we obtain the following Caccioppoli-type inequality

$$\operatorname*{ess\,sup}_{t\in(t_0-r^2,t_0)}\int\limits_{B_r}\theta^4|\boldsymbol{B}-\boldsymbol{B}_{r,X_0}|^2dx+\int\limits_{Q_r}\theta^4|\nabla\boldsymbol{B}|^2dxdt$$
$$\le\frac{c}{(r-\rho)^2}\int\limits_{Q_r}|\boldsymbol{B}-\boldsymbol{B}_{r,X_0}|^2dxdt+c\int\limits_{Q_r}|\boldsymbol{g}|^2+\theta^4|\boldsymbol{B}|^2|\boldsymbol{u}|^2dxdt$$
$$+\frac{c}{r-\rho}\int\limits_{Q_r}\theta^3|\nabla\boldsymbol{B}|\,|\boldsymbol{B}|\,|\boldsymbol{B}-\boldsymbol{B}_{r,X_0}|dxdt$$
$$=\frac{c}{(r-\rho)^2}\int\limits_{Q_r}|\boldsymbol{B}-\boldsymbol{B}_{r,X_0}|^2dxdt+c\int\limits_{Q_r}|\boldsymbol{g}|^2+\theta^4|\boldsymbol{B}|^2|\boldsymbol{u}|^2dxdt+J. \tag{2.7}$$

Let $\gamma \in [-3, 3]$. Applying Hölder's and Young's inequality, we estimate

$$
\begin{aligned}
J \leq{}& \frac{c}{(r-\rho)^2} |\boldsymbol{B}_{r,X_0}|^2 \int\limits_{Q_r} |\boldsymbol{B} - \boldsymbol{B}_{r,X_0}|^2 dxdt \\
&+ \frac{c}{r-\rho} \bigg(\int\limits_{Q_r} \theta^{3+\gamma} |\boldsymbol{B} - \boldsymbol{B}_{r,X_0}|^4 dxdt \bigg)^{1/2} \bigg(\int\limits_{Q_r} \theta^{3-\gamma} |\nabla \boldsymbol{B}|^2 dxdt \bigg)^{1/2} \\
&+ \frac{1}{2} \int\limits_{Q_r} \theta^4 |\nabla \boldsymbol{B}|^2 dxdt.
\end{aligned}
$$

Inserting the estimate of J into (2.7), we are led to

$$
\begin{aligned}
&\operatorname*{ess\,sup}_{t \in (t_0 - r^2, t_0)} \int\limits_{B_r} \theta^4 |\boldsymbol{B} - \boldsymbol{B}_{r,X_0}|^2 dx + \int\limits_{Q_r} \theta^4 |\nabla \boldsymbol{B}|^2 dxdt \\
&\quad \leq \frac{c}{(r-\rho)^2} (1 + |\boldsymbol{B}_{r,X_0}|^2) \int\limits_{Q_r} |\boldsymbol{B} - \boldsymbol{B}_{r,X_0}|^2 dxdt \\
&\qquad + \frac{c}{r-\rho} \bigg(\int\limits_{Q_r} \theta^{3+\gamma} |\boldsymbol{B} - \boldsymbol{B}_{r,X_0}|^4 dxdt \bigg)^{1/2} \bigg(\int\limits_{Q_r} \theta^{3-\gamma} |\nabla \boldsymbol{B}|^2 dxdt \bigg)^{1/2} \\
&\qquad + c \int\limits_{Q_r} (|\boldsymbol{g}|^2 + \theta^4 |\boldsymbol{B}|^2 |\boldsymbol{u}|^2) dxdt. \qquad (2.8)
\end{aligned}
$$

This proves (2.5). On the other hand, by means of Sobolev's embedding theorem we get

$$
\begin{aligned}
&\fint_{Q_r} \theta^4 |\boldsymbol{B} - \boldsymbol{B}_{r,X_0}|^4 dxdt \\
&\leq c r^{-4} \|\theta^2 (\boldsymbol{B} - \boldsymbol{B}_{r,X_0})\|^2_{L^\infty(t_0 - r^2, t_0; L^2(B_r))} \|\nabla \boldsymbol{B}\|^2_{L^2(Q_r)} \\
&\quad + c r^{-4} (r-\rho)^{-2} \|\theta^2 (\boldsymbol{B} - \boldsymbol{B}_{r,X_0})\|^2_{L^\infty(t_0 - r^2, t_0; L^2(B_r))} \|\boldsymbol{B} - \boldsymbol{B}_{r,X_0}\|^2_{L^2(Q_r)} \\
&\leq \frac{c}{(r-\rho)^2} \|\theta^2 (\boldsymbol{B} - \boldsymbol{B}_{r,X_0})\|^2_{L^\infty(t_0 - r^2, t_0; L^2(B_r))} (F(r)^2 + G(r)^2). \qquad (2.9)
\end{aligned}
$$

Combining (2.8) with $\gamma = 1$ and (2.9) with help of Young's inequality, we get

$$
\begin{aligned}
&\fint_{Q_r} \theta^4 |\boldsymbol{B} - \boldsymbol{B}_{r,X_0}|^4 dxdt \\
&\quad \leq \frac{c r^4}{(r-\rho)^4} (1 + |\boldsymbol{B}_{r,X_0}|^2) G(r)^2 (F(r)^2 + G(r)^2) \\
&\qquad + \frac{c r^6}{(r-\rho)^6} (F(r)^6 + G(r)^6) \\
&\qquad + \frac{c}{(r-\rho)^2} \int\limits_{Q_r} (|\boldsymbol{g}|^2 + \theta^4 |\boldsymbol{B}|^2 |\boldsymbol{u}|^2) dxdt (F(r)^2 + G(r)^2). \qquad (2.10)
\end{aligned}
$$

Estimating $|\boldsymbol{B}|^2 \leq 2|\boldsymbol{B} - \boldsymbol{B}_{r,X_0}|^2 + 2|\boldsymbol{B}_{r,X_0}|^2$ and applying Young's inequality, we obtain (2.6). Thus, the proof of the Lemma is complete. $\blacksquare$

Remark 2.4. From (2.5) with $\gamma = -1$ along with Young's inequality we get

$$\begin{aligned}
&\bigg(\frac{1}{r^2}\operatorname*{ess\,sup}_{t\in(t_0-\rho^2,t_0)}\int\limits_{B_\rho}|\boldsymbol{B}(t)-\boldsymbol{B}_{r,X_0}|^2dx\bigg)^{1/2}+F(\rho)\\
&\qquad\le\frac{cr}{r-\rho}\Big\{(1+|\boldsymbol{B}_{r,X_0}|)E(r)+E(r)^2\Big\}\\
&\qquad\qquad+\frac{c}{\rho}\Big\{\|\boldsymbol{u}\|_{2,Q_r}(E(r)+|\boldsymbol{B}_{r,X_0}|)+\|\boldsymbol{g}\|_{2,Q_r}\Big\}. \qquad (2.11)
\end{aligned}$$

Furthermore, using the parabolic Poincaré-type inequality (cf. Lemma A.1, appendix below), we find

$$\begin{aligned}
&\fint_{Q_r}|\boldsymbol{B}-\boldsymbol{B}_{r,X_0}|^2dxdt\\
&\qquad\le c(1+|\boldsymbol{B}_{r,X_0}|^2)r^{-2}\int\limits_{Q_r}|\nabla\boldsymbol{B}|^2dxdt\\
&\qquad\quad+c(1+|\boldsymbol{B}_{r,X_0}|^2)r^{-2}\int\limits_{Q_r}(|\boldsymbol{g}|^2+|\boldsymbol{u}|^2)dxdt\\
&\qquad\quad+C_1r^{-2}\int\limits_{Q_r}(|\nabla\boldsymbol{B}|^2+|\boldsymbol{u}|^2)dxdt\fint_{Q_r}|\boldsymbol{B}-\boldsymbol{B}_{r,X_0}|^2dxdt \qquad (2.12)
\end{aligned}$$

with an absolute constant $C_1>0$. Thus, assuming that

$$C_1\bigg\{r^{-2}\int\limits_{Q_r}|\nabla\boldsymbol{B}|^2dxdt+4\bigg(\int\limits_{Q_r}|\boldsymbol{u}|^4dxdt\bigg)^{1/2}\bigg\}\le\frac{1}{2}, \qquad (2.13)$$

(2.12) leads to

$$G(r)\le c(1+|\boldsymbol{B}_{r,X_0}|)(F(r)+H(r)), \qquad (2.14)$$

where

$$H(r)=H(r,X_0)=r^{-1}\|\boldsymbol{g}\|_{2,Q_r}+\|\boldsymbol{u}\|_{4,Q_r},\quad 0<r<\sqrt{t_0}.$$

Substituting $G(r)$ on the right of (2.6) by (2.14), setting $\rho=\frac{r}{2}$ therein, we arrive at

$$E(r/2)\le C_2(1+|\boldsymbol{B}_{r,X_0}|^2)\Big\{F(r)+F(r)^2+H(r)+H(r)^2\Big\} \qquad (2.15)$$

with an absolute constant $C_2>0$, provided (2.13) is fulfilled.

From (2.11) with $\rho=\frac{r}{2}$ we deduce that

$$F(r/2)\le C_3(1+|\boldsymbol{B}_{r,X_0}|)\Big\{E(r)+E(r)^2+H(r)+H(r)^2\Big\} \qquad (2.16)$$

with an absolute constant $C_3>0$.

3 Blow-up lemma

In what follows we define the space

$$V^2(Q_r) = L^\infty(t_0 - r^2, t_0; W^{1,2}(B_r(x_0))) \cap L^\infty(t_0 - r^2, t_0; L^2(B_r(x_0)))$$

for $X_0 = (x_0, t_0)$ and $0 < r < +\infty$.

We begin our discussion with the following fundamental estimate for solutions to the model problem in $Q_1 = Q_1(0,0)$, which will be used in the blow-up lemma below.

Lemma 3.1. *Let $\boldsymbol{\Lambda} \in \mathbb{R}^3$. Let $\boldsymbol{W} \in L^4(Q_1)$ such that $\boldsymbol{W}|_{Q_\sigma} \in V^2(Q_\sigma)$ for all $0 < \sigma < 1$ solves*

$$\partial_t \boldsymbol{W} - \Delta \boldsymbol{W} = -\nabla \times ((\nabla \times \boldsymbol{W}) \times \boldsymbol{\Lambda}) \quad \text{in} \quad Q_1 \tag{3.1}$$

in sense of distributions, i. e.

$$\int\limits_{B_1} \boldsymbol{W}(t) \cdot \boldsymbol{\Phi}(t) dx + \int\limits_{-1}^{t} \int\limits_{B_1} (-\boldsymbol{W} \cdot \partial_t \boldsymbol{\Phi} + \nabla \boldsymbol{W} : \nabla \boldsymbol{\Phi}) dx ds$$
$$= -\int\limits_{-1}^{t} \int\limits_{B_1} ((\nabla \times \boldsymbol{W}) \times \boldsymbol{\Lambda}) \cdot \nabla \times \boldsymbol{\Phi} dx ds \tag{3.2}$$

for all $\boldsymbol{\Phi} \in W^{1,2}(Q_1)$ compactly supported in Q_1, for a. e. $t \in (-1, 0)$. Then,

$$\bigg(\fint_{Q_\tau} |\boldsymbol{W} - \boldsymbol{W}_{Q_\tau}|^4 dx dt \bigg)^{1/4} \le C_0 \tau (1 + |\boldsymbol{\Lambda}|^5) \bigg(\fint_{Q_1} |\boldsymbol{W} - \boldsymbol{W}_{Q_1}|^4 dx dt \bigg)^{1/4} \tag{3.3}$$

for all $0 < \tau < 1$, where $C_0 > 0$ denotes a universal constant.

Proof Since the assertion is trivial for $\frac{1}{4} < \tau < 1$, we may assume that $0 < \tau \le \frac{1}{4}$. Let $\zeta \in C^\infty_{\mathrm{c}}(\mathbb{R}^3)$ be a suitable cut-off function for Q_τ and $Q_{1/2}$. Inserting the admissible test function $\boldsymbol{\Phi} = \zeta^{2m}(\boldsymbol{W} - \boldsymbol{W}_{B_1})$ $(m \in \mathbb{N})$ into (3.2), by using Cauchy-Schwarz's inequality along with Young's inequality, we are led to

$$\operatorname*{ess\,sup}_{t \in (-1,0)} \int\limits_{B_1} \zeta^{2m} |\boldsymbol{W}(t)|^2 dx + \int\limits_{Q_1} \zeta^{2m} |\nabla \boldsymbol{W}|^2 dx dt$$
$$\le c(1 + |\boldsymbol{\Lambda}|^2) \int\limits_{Q_1} \zeta^{2m-2} |\boldsymbol{W} - \boldsymbol{W}_{Q_1}|^2 dx dt. \tag{3.4}$$

If $\boldsymbol{W}$ is smooth in Q_1, since (3.1) is a linear system, the same inequality holds true for $D^\alpha \boldsymbol{W}$ in place of $\boldsymbol{W}$ for any multi-index α. By a standard mollifying argument together with Sobolev's embedding theorem we see that $\boldsymbol{W}$ is smooth in Q_1. By an iterative application of (3.4) with $m = 4, 3, 2, 1$ we obtain

$$\operatorname*{ess\,sup}_{t \in (-1,0)} \int\limits_{B_1} \zeta^8 |D^\alpha \boldsymbol{W}|^2 dx \le c(1 + |\boldsymbol{\Lambda}|^8) \int\limits_{Q_1} |\boldsymbol{W} - \boldsymbol{W}_{Q_1}|^2 dx dt \quad \forall\, |\alpha| \le 3. \tag{3.5}$$

By means of Sobolev's embedding theorem and Jensen's inequality we get

$$\|\nabla \boldsymbol{W}\|^4_{\infty,Q_{1/2}} \le c(1+|\boldsymbol{\Lambda}|^{16}) \int\limits_{Q_1} |\boldsymbol{W} - \boldsymbol{W}_{Q_1}|^4 dxdt. \tag{3.6}$$

Applying Poincaré's inequality, we arrive at

$$⨍_{Q_\tau} |\boldsymbol{W} - \boldsymbol{W}_{Q_\tau}|^4 dxdt \le c\tau^4(1+|\Lambda|^4)\|\nabla \boldsymbol{W}\|^4_{\infty,Q_{1/2}}. \tag{3.7}$$

Combination of (3.6) and (3.7) gives the desired estimate. ■

In our discussion below we make use of the notion of the Morrey space. Let $K \subset Q$ be a compact set. Define, $d_K = \min\{t \in (0,T) \,|\, t \in K\}$. We say f belongs to the Morrey space $\mathcal{M}^{p,\lambda}(K)$ if

$$[f]_{\mathcal{M}^{p,\lambda},K} := \sup\left\{ r^{-\lambda} \int\limits_{Q_r(X_0)} |f|^p dxdt \,\middle|\, X_0 \in K, 0 < r \le d_K \right\} < +\infty.$$

Furthermore, by $f \in \mathcal{M}^{p,\lambda}_{\mathrm{loc}}(Q)$ we mean $f \in \mathcal{M}^{p,\lambda}(K)$ for all compact set $K \subset Q$.

Now we are ready to state the following key lemma.

Lemma 3.2. *Let $\boldsymbol{g} \in \mathcal{M}^{2,\lambda}_{\mathrm{loc}}(Q)$ for some $2 < \lambda < 4$. For every $0 < \tau < \frac{1}{2}, 0 < M, L < +\infty$, compact set $K \subset Q$ and $0 < \alpha < \frac{\lambda-2}{2}$, there exist positive numbers $\varepsilon_0 = \varepsilon_0(\tau, M, L, K, \alpha)$, $R_0 = R_0(\tau, M, L, K, \alpha) < d_K$ and $\delta_0 = \delta_0(\tau, M, L, K, \alpha) \le 1$ such that, if $\boldsymbol{B} \in V^2(Q)$ is a weak solution to* (2.1)*with $0 < \delta \le \delta_0$ and $\boldsymbol{u} \in L^{8/(4-\lambda)}(Q)$ such that*

$$\|\boldsymbol{u}\|_{8/(4-\lambda),Q} \le L, \tag{3.8}$$

and if for $X_0 \in K$ and $0 < R \le R_0$ the following condition is fulfilled

$$|\boldsymbol{B}_{R,X_0}| \le M, \quad E(R, X_0) + R^\alpha \le \varepsilon_0, \tag{3.9}$$

then there holds

$$E(\tau R, X_0) \le 2\tau C_0(1+M^5)(E(R,X_0)+R^\alpha), \tag{3.10}$$

where $C_0 > 0$ stands for the constant appearing on the right hand side of (3.3).

Proof Assume the assertion of the Lemma is not true. Then there exist $0 < \tau < \frac{1}{2}, 0 < M, L < +\infty$, a compact set $K \subset \Omega$ and $0 < \alpha < \frac{\lambda-2}{2}$ as well as sequences $\{\varepsilon_k\}, \{\delta_k\} \subset (0,1)$ with $\varepsilon_k, \delta_k \to 0$ as $k \to +\infty$, $\{R_k\} \subset (0, d_K)$, $\{X_k\} = \{(x_k, t_k)\} \subset K$, $\{\boldsymbol{u}^{(k)}\} \subset L^{8/(4-\lambda)}(Q)$ fulfilling

$$\|\boldsymbol{u}^{(k)}\|_{8/(4-\lambda)} \le L \quad \forall\, k \in \mathbb{N}, \tag{3.11}$$

and a sequence $\{\boldsymbol{B}^{(k)}\} \subset V^2(Q)$, being a weak solutions to (2.1) replacing $\boldsymbol{u}$ by $\boldsymbol{u}^{(k)}$ and δ by δ_k respectively, such that

$$|\boldsymbol{B}^{(k)}_{R_k,X_k}| \le M, \quad E_k(R_k, X_k) + R_k^\alpha = \varepsilon_k \tag{3.12}$$

and

$$E_k(\tau R_k, X_k) > 2\tau C_0(1+M^5)(E_k(R_k, X_k) + R_k^\alpha). \tag{3.13}$$

Here we have used the notation

$$E_k(r, X_k) = \left(\fint_{Q_r(X_k)} |\boldsymbol{B}^{(k)} - \boldsymbol{B}^{(k)}_{r,X_k}|^4 dxdt \right)^{1/4}, \quad X_k \in K, 0 < r \le d_K$$

($k \in \mathbb{N}$). Note that (3.12) yields $R_k \to 0$ as $k \to +\infty$.

Next, for $Y := (y, s) \in Q_1(0)$ we define

$$\begin{aligned}
\boldsymbol{W}_k(Y) &= \frac{1}{\varepsilon_k}(\boldsymbol{B}^{(k)}(x_k + R_k y, t_k + R_k^2 s) - \boldsymbol{B}^{(k)}_{R_k,X_k}),\\
\boldsymbol{v}_k(Y) &= \boldsymbol{u}^{(k)}(x_k + R_k y, t_k + R_k^2 s),\\
\boldsymbol{g}_k(Y) &= \boldsymbol{g}(x_k + R_k y, t_k + R_k^2 s),
\end{aligned}$$

($k \in \mathbb{N}$). Furthermore, we set

$$\mathscr{E}_k(\sigma) = \left(\fint_{Q_\sigma} |\boldsymbol{W}_k - (\boldsymbol{W}_k)_{Q_\sigma}|^4 dyds \right)^{1/4}, \quad 0 < \sigma \le 1.$$

Then (3.12) and (3.13) turn into

$$|\boldsymbol{B}^{(k)}_{R_k,X_k}| \le M, \quad \mathscr{E}_k(1) + \frac{R_k^\alpha}{\varepsilon_k} = 1, \tag{3.14}$$

and

$$\mathscr{E}_k(\tau) > 2\tau C_0(1+M^5)\Big(\mathscr{E}_k(1) + \frac{R_k^\alpha}{\varepsilon_k}\Big) = 2\tau C_0(1+M^5) \tag{3.15}$$

respectively.

Using the chain rule, restriction of system (2.1) to $Q_{R_k}(X_k)$ takes the form

$$\begin{aligned}
&\partial_t \boldsymbol{W}_k - \Delta \boldsymbol{W}_k\\
&\quad = -\nabla \times \left((\nabla \times \boldsymbol{W}_k) \times \frac{\varepsilon_k \boldsymbol{W}_k + \boldsymbol{B}^{(k)}_{R_k,X_k}}{1 + \delta_k |\varepsilon_k \boldsymbol{W}_k + \boldsymbol{B}^{(k)}_{R_k,X_k}|} \right)\\
&\qquad + \frac{R_k}{\varepsilon_k} \nabla \times \left(\boldsymbol{v}_k \times \frac{\varepsilon_k \boldsymbol{W}_k + \boldsymbol{B}^{(k)}_{R_k,X_k}}{1 + \delta_k |\varepsilon_k \boldsymbol{W}_k + \boldsymbol{B}^{(k)}_{R_k,X_k}|} \right) + \frac{R_k}{\varepsilon_k} \nabla \times \boldsymbol{g}_k
\end{aligned} \tag{3.16}$$

in Q_1. Thus, $\boldsymbol{W}_k \in V^2(Q_1)$ is a weak solution to (3.16).

Let $0 < \sigma < 1$. Using the transformation formula, noticing that $|\boldsymbol{B}^{(k)}_{R_k,X_k}| \le M$, the Caccioppoli-type inequality (2.11) with $r = R_k$ and $\rho = \sigma R_k$ turns into

$$\begin{aligned}
&\|\boldsymbol{W}_k\|_{L^\infty(-\sigma^2,0;L^2(B_\sigma))} + \|\nabla \boldsymbol{W}_k\|_{2,B_\sigma}\\
&\quad \le c(1-\sigma)^{-1}\Big((1+M)\mathscr{E}_k(1) + \varepsilon_k \mathscr{E}_k(1)^2\Big)\\
&\qquad + \frac{cR_k^{-1}}{\varepsilon_k}\Big(\|\boldsymbol{u}^{(k)}\|_{2,Q_{R_k}(X_k)}(\varepsilon_k \mathscr{E}_k(1) + M) + \|\boldsymbol{g}\|_{2,Q_{R_k}(X_k)}\Big).
\end{aligned} \tag{3.17}$$

As $\boldsymbol{g}_k \in \mathcal{M}^{2,\lambda}(K)$ observing (3.14), we see that

$$\frac{R_k^{-1}}{\varepsilon_k}\|\boldsymbol{g}\|_{2,Q_{R_k}(x_k)} \le \frac{R_k^{(\lambda-2)/2}}{\varepsilon_k}[\boldsymbol{g}]_{\mathcal{M}^{2,\lambda}(K)} \le R_k^{(\lambda-2)/2-\alpha}[\boldsymbol{g}]_{\mathcal{M}^{2,\lambda}(K)}. \tag{3.18}$$

Similarly, by (3.11) and (3.14) we get

$$\frac{R_k^{-1}}{\varepsilon_k}\|\boldsymbol{u}^{(k)}\|_{2,Q_{R_k}(x_k)} \le c\frac{R_k^{(\lambda-2)/2}}{\varepsilon_k}\|\boldsymbol{u}^{(k)}\|_{8/4-\lambda,Q} \le cR_k^{(\lambda-2)/2-\alpha}L. \tag{3.19}$$

Thus, from (3.17) with help of (3.18), (3.19) and (3.14) we obtain

$$\|\boldsymbol{W}_k\|_{L^\infty(-\sigma^2,0;L^2(B_\sigma))} + \|\nabla\boldsymbol{W}_k\|_{2,Q_\sigma} \le c(1-\sigma)^{-1}(M+1) + c([\boldsymbol{g}]_{\mathcal{M}^{2,\lambda}(K)} + L). \tag{3.20}$$

In addition, in view of (3.14) we estimate

$$\|\boldsymbol{W}_k\|_{4,Q_1} = (\operatorname{mes} B_1)^{1/4}\mathscr{E}_k(1) \le (\operatorname{mes} B_1)^{1/4}. \tag{3.21}$$

From (3.20) and (3.21) it follows that $\{\boldsymbol{W}_k\}$ is bounded in $V^2(Q_\sigma)$ for all $0<\sigma<1$ and bounded in $L^4(Q_1)$. Thus, by means of reflexivity, eventually passing to subsequences, we get $\boldsymbol{W} \in L^4(Q_1)$ with $\boldsymbol{W} \in V^2(Q_\sigma)$ for all $0<\sigma<1$ and $\boldsymbol{\Lambda} \in \mathbb{R}^3$ such that

$$\boldsymbol{W}_k \to \boldsymbol{W} \quad \textit{weakly in} \quad L^4(Q_1) \quad \text{as} \quad k \to +\infty, \tag{3.22}$$

$$\nabla\boldsymbol{W}_k \to \nabla\boldsymbol{W} \quad \textit{weakly in} \quad L^2(Q_\sigma) \quad \text{as} \quad k \to +\infty \quad \forall\, 0<\sigma<1, \tag{3.23}$$

$$\boldsymbol{W}_k \to \boldsymbol{W} \quad \textit{weakly* in} \quad L^\infty(-\sigma^2,0;L^2(B_\sigma)) \quad \text{as} \quad k \to +\infty \quad \forall\, 0<\sigma<1, \tag{3.24}$$

$$\boldsymbol{\Lambda}_k \to \boldsymbol{\Lambda} \quad \text{in} \quad \mathbb{R}^3 \quad \text{as} \quad k \to +\infty. \tag{3.25}$$

On the other hand, from (3.16) we deduce that the sequence of distributive time derivative $\{\boldsymbol{W}_k'\}$ is bounded in $L^{4/3}(-\sigma^2,0;W^{-1,4/3}(B_\sigma))$. From this fact together with (3.22) it follows that

$$\boldsymbol{W}_k \to \boldsymbol{W} \quad \textit{strongly in} \quad L^2(Q_\sigma) \quad \text{as} \quad k \to +\infty \quad \forall\, 0<\sigma<1. \tag{3.26}$$

Thus, we are in a position to carry out the passage to the limit $k \to +\infty$ in the weak formulation of (3.16) to deduce that $\boldsymbol{W}$ is a weak solution to the linear system (3.1).

Our next aim is to prove the strong convergence of $\boldsymbol{W}_k \to \boldsymbol{W}$ in $L^4(Q_\sigma)$ $(0<\sigma<1)$. We first state the following energy equality,

$$\begin{aligned}
&\frac{1}{2}\int\limits_{B_1}\phi^2(t)|\boldsymbol{W}_k(t)|^2dy + \int\limits_{-1}^{t}\int\limits_{B_1}\phi^2|\nabla\boldsymbol{W}_k|^2dyds \\
&= \frac{1}{2}\int\limits_{-1}^{t}\int\limits_{B_1}(\partial_t\phi^2 + \Delta\phi^2)|\boldsymbol{W}_k|^2dyds \\
&\quad + \int\limits_{-1}^{t}\int\limits_{B_1}(\nabla\times\boldsymbol{W}_k)\times\frac{\varepsilon_k\boldsymbol{W}_k + \boldsymbol{B}^{(k)}_{R_k,X_k}}{1+\delta_k|\varepsilon_k\boldsymbol{W}_k + \boldsymbol{B}^{(k)}_{R_k,X_k}|}\cdot(\boldsymbol{W}_k\times\nabla\phi^2)dyds \\
&\quad + \frac{R_k}{\varepsilon_k}\int\limits_{-1}^{t}\int\limits_{B_1}\left\{\boldsymbol{v}_k\times\frac{\varepsilon_k\boldsymbol{W}_k + \boldsymbol{B}^{(k)}_{R_k,X_k}}{1+\delta_k|\varepsilon_k\boldsymbol{W}_k + \boldsymbol{B}^{(k)}_{R_k,X_k}|} + \boldsymbol{g}_k\right\}\nabla\times(\phi^2\boldsymbol{W}_k)dyds
\end{aligned} \tag{3.27}$$

for all $t \in [-1, 0]$. In view of (3.22), (3.23), (3.25) and (3.26) on both sides of (3.27) with $t = 0$ letting $k \to +\infty$, we infer

$$\lim_{k\to\infty} \bigg(\frac{1}{2}\int\limits_{B_1} \phi^2(0)|\boldsymbol{W}_k(0)|^2 dy + \int\limits_{Q_1} \phi^2 |\nabla \boldsymbol{W}_k|^2 dyds\bigg)$$

$$= \frac{1}{2}\int\limits_{Q_1} (\partial_t \phi^2 + \Delta\phi^2)|\boldsymbol{W}|^2 dyds - \int\limits_{Q_1} (\nabla \times \boldsymbol{W}) \times \boldsymbol{\Lambda} \cdot (\boldsymbol{W} \times \nabla\phi^2) dyds. \tag{3.28}$$

Since $\boldsymbol{W}$ is a weak solution to (3.1), there holds

$$\frac{1}{2}\int\limits_{B_1} \phi^2(0)|\boldsymbol{W}(0)|^2 dy + \int\limits_{Q_1} \phi^2 |\nabla \boldsymbol{W}|^2 dyds$$

$$= \frac{1}{2}\int\limits_{Q_1} (\partial_t \phi^2 + \Delta\phi^2)|\boldsymbol{W}|^2 dyds - \int\limits_{Q_1} (\nabla \times \boldsymbol{W}) \times \boldsymbol{\Lambda} \cdot (\boldsymbol{W} \times \nabla\phi^2) dyds. \tag{3.29}$$

Noticing that

$$\begin{cases} (\phi(0)\boldsymbol{W}_k(0), \phi\nabla\boldsymbol{W}_k) \to (\phi(0)\boldsymbol{W}(0), \phi\nabla\boldsymbol{W}) \\ \textit{weakly in} \quad L^2(B_1) \times L^2(Q_1) \quad \text{as} \quad k \to +\infty \end{cases}$$

from (3.28) and (3.29), we deduce that

$$\nabla \boldsymbol{W}_k \to \nabla \boldsymbol{W} \quad \textit{strongly in} \quad L^2(Q_\sigma) \quad \text{as} \quad k \to +\infty \quad \forall\, 0 < \sigma < 1.$$

Accordingly,

$$\lim_{k\to\infty} \mathscr{E}_k(\sigma) = \mathscr{E}(\sigma) \quad \forall\, 0 < \sigma < 1, \tag{3.30}$$

where $\mathscr{E}(\sigma) = \bigg(\displaystyle⨍_{B_\sigma} |\boldsymbol{W} - \boldsymbol{W}_{B_\sigma}|^4 dy\bigg)^{1/2}$. In particular, thanks to (3.30) (with $\sigma = \tau$) from (3.15) we get

$$\mathscr{E}(\tau) \ge 2\tau C_0(1 + M^5). \tag{3.31}$$

Since $\boldsymbol{W}$ is a weak solution to (3.1) and $|\boldsymbol{\Lambda}| \le M$, appealing to Lemma 3.1, we find

$$\mathscr{E}(\tau) \le \tau C_0(1 + M^5)\mathscr{E}(1). \tag{3.32}$$

On the other hand, by virtue of the lower semi continuity of the norm together with (3.15) and (3.30) we get

$$\begin{aligned} \mathscr{E}(1) &\le \liminf_{k\to\infty} \Big(\mathscr{E}_k(1) + \frac{R_k^\alpha}{\varepsilon_k}\Big) \le \frac{1}{2\tau C_0(1 + M^5)} \lim_{k\to\infty} \mathscr{E}_k(\tau) \\ &= \frac{1}{2\tau C_0(1 + M^5)} \mathscr{E}(\tau). \end{aligned}$$

Estimating the right of (3.32) by the inequality, we have just obtained we are led to $\mathscr{E}(\tau) \le \frac{1}{2}\mathscr{E}(\tau)$ and hence $\mathscr{E}(\tau) = 0$, which contradicts to (3.31). Whence, the assumption cannot be true, which completes the proof of the Lemma. ■

4 Construction of approximate solutions

The aim of the present section is to construct a weak solution of the Hall-MHD system (1.1)–(1.5) as a limit of a sequence of solutions to the a corresponding approximate system. As we will see in the following section, such solution will satisfy the desired partial regularity as stated in the main result of the present paper.

Let $\{\delta_m\} \subset (0,1) \quad (m \in \mathbb{N})$ be a sequence, such that $\delta_m \to 0$ as $m \to +\infty$. Now, we consider the following approximate system

$$\begin{aligned} \partial_t \boldsymbol{u}_m + & \frac{\boldsymbol{\omega}_m}{1+\delta_m|\boldsymbol{B}_m|} \times \boldsymbol{u}_m - \Delta \boldsymbol{u}_m \\ & = -\nabla p_m + (\nabla \times \boldsymbol{B}_m) \times \frac{\boldsymbol{B}_m}{1+\delta_m|\boldsymbol{B}_m|} + \boldsymbol{f}, \end{aligned} \tag{4.1}$$

$$\begin{aligned} \partial_t \boldsymbol{B}_m + & \nabla \times \left(\frac{\boldsymbol{B}_m}{1+\delta_m|\boldsymbol{B}_m|} \times \boldsymbol{u}_m \right) - \Delta \boldsymbol{B}_m \\ & = -\nabla \times \left(\nabla \times \boldsymbol{B}_m \times \frac{\boldsymbol{B}_m}{1+\delta_m|\boldsymbol{B}_m|} \right) + \nabla \times \boldsymbol{g}, \end{aligned} \tag{4.2}$$

$$\nabla \cdot \boldsymbol{u}_m = 0, \quad \nabla \cdot \boldsymbol{B}_m = 0, \tag{4.3}$$

in $Q = \mathbb{R}^2 \times (0,T)$, together with the initial condition

$$\boldsymbol{u}_m = \boldsymbol{u}_0, \qquad \boldsymbol{B}_m = \boldsymbol{B}_0, \quad \text{in} \quad \mathbb{R}^2 \times \{0\}. \tag{4.4}$$

Here $(\boldsymbol{u}_m, p_m, \boldsymbol{B}_m) \in V^2_{\mathrm{div}}(Q) \times L^2(Q) \times V^2_{\mathrm{div}}(Q)$ is called a weak solution to (4.1)-(4.3) if

$$\begin{aligned} & \int_Q (-\boldsymbol{u}_m \cdot \partial_t \boldsymbol{\varphi} + \nabla \boldsymbol{u}_m : \nabla \boldsymbol{\varphi} - \boldsymbol{u}_m \otimes \boldsymbol{u}_m : \nabla \boldsymbol{\varphi}) dxdt \\ & = \int_Q p_m \nabla \cdot \boldsymbol{\varphi} dxdt + \int_Q \Big((\nabla \times \boldsymbol{B}_m) \times \frac{\boldsymbol{B}_m}{1+\delta_m|\boldsymbol{B}_m|} \Big) \cdot \boldsymbol{\varphi} dxdt + \int_Q \boldsymbol{f} \cdot \boldsymbol{\varphi} dxdt, \end{aligned} \tag{4.5}$$

$$\begin{aligned} & \int_Q (-\boldsymbol{B}_m \cdot \partial_t \boldsymbol{\varphi} + \nabla \boldsymbol{B}_m : \nabla \boldsymbol{\varphi}) dxdt \\ & = -\int_Q \Big((\nabla \times \boldsymbol{B}_m - \boldsymbol{u}_m) \times \frac{\boldsymbol{B}_m}{1+\delta_m|\boldsymbol{B}_m|} \Big) \cdot \nabla \times \boldsymbol{\varphi} dx + \int_Q \boldsymbol{g} \cdot \nabla \times \boldsymbol{\varphi} dxdt \end{aligned} \tag{4.6}$$

for all $\boldsymbol{\varphi} \in C^\infty_{\mathrm{c}}(Q)$.

The existence of weak solutions to (4.1)–(4.4) is given by the following

Lemma 4.1. *Let $\boldsymbol{u}_0 \in L^2_{\mathrm{div}}, \boldsymbol{B}_0 \in L^2$ and $\boldsymbol{f}, \boldsymbol{g} \in L^2(Q)$. Then for every $m \in \mathbb{N}$ there exists a weak solution $(\boldsymbol{u}_m, p_m, \boldsymbol{B}_m) \in V^2_{\mathrm{div}}(Q) \times L^2(0,T; L^2_{\mathrm{loc}}) \times V^2_{\mathrm{div}}(Q)$ to (4.1)–(4.4), such that*

$$\nabla \boldsymbol{u}_m \in V^2(Q_r), \quad \forall \overline{Q}_r \subset Q. \tag{4.7}$$

Furthermore, this solution fulfills the energy equality

$$
\begin{aligned}
&\frac{1}{2}\|\boldsymbol{u}_m(t)\|_2^2 + \frac{1}{2}\|\boldsymbol{B}_m(t)\|_2^2 + \int\limits_0^t (\|\nabla \boldsymbol{u}_m(s)\|_2^2 + \|\nabla \boldsymbol{B}_m(s)\|_2^2) ds \\
&\qquad = \frac{1}{2}\|\boldsymbol{u}_0\|_2^2 + \frac{1}{2}\|\boldsymbol{B}_0\|_2^2 + \int\limits_0^t \int\limits_{\mathbb{R}^2} (\boldsymbol{f}\cdot \boldsymbol{u}_m + \boldsymbol{g}\cdot \nabla\times \boldsymbol{B}_m) dx ds
\end{aligned}
\tag{4.8}
$$

for a. e. $t \in (0,T)$.

Proof Let $m \in \mathbb{N}$ be fixed. Let $\beta_l \to 0^+$ as $l \to +\infty$. By using the well-known monotone operator theory we get a weak solution $(\boldsymbol{u}_{m,l}, p_{m,l}, \boldsymbol{B}_{m,l}) \in V^2_{\mathrm{div}}(Q) \times L^2(0,T; L^2_{\mathrm{loc}}) \times V^2_{\mathrm{div}}(Q)$ of the following approximate system

$$
\begin{aligned}
&\partial_t \boldsymbol{u}_{m,l} + \frac{\boldsymbol{\omega}_{m,l}}{1+\delta_m|\boldsymbol{B}_{m,l}| + \beta_l |\boldsymbol{V}_{m,l}|} \times \boldsymbol{u}_{m,l} - \Delta \boldsymbol{u}_{m,l} \\
&\qquad = -\nabla p_{m,l} + (\nabla \times \boldsymbol{B}_{m,l}) \times \frac{\boldsymbol{B}_{m,l}}{1+\delta_m|\boldsymbol{B}_{m,l}| + \beta_l |\boldsymbol{V}_{m,l}|} + \boldsymbol{f},
\end{aligned}
\tag{4.9}
$$

$$
\begin{aligned}
&\partial_t \boldsymbol{B}_{m,l} + \nabla \times \frac{\boldsymbol{B}_{m,l}}{1+\delta_m|\boldsymbol{B}_{m,l}| + \beta_l |\boldsymbol{V}_{m,l}|} \times \boldsymbol{u}_{m,l} - \Delta \boldsymbol{B}_{m,l} \\
&\qquad = -\nabla \times \Big(\nabla \times \boldsymbol{B}_{m,l} \times \frac{\boldsymbol{B}_{m,l}}{1+\delta_m|\boldsymbol{B}_{m,l}| + \beta_l |\boldsymbol{V}_{m,l}|}\Big) + \nabla \times \boldsymbol{g}
\end{aligned}
\tag{4.10}
$$

$$
\nabla \cdot \boldsymbol{u}_{m,l} = 0, \qquad \nabla \cdot \boldsymbol{B}_{m,l} = 0 \tag{4.11}
$$

in $Q = \mathbb{R}^2 \times (0,T)$ together with the initial condition

$$
\boldsymbol{u}_{m,l} = \boldsymbol{u}_0, \qquad \boldsymbol{B}_{m,l} = \boldsymbol{B}_0, \quad \text{in} \quad \mathbb{R}^2 \times \{0\}, \tag{4.12}
$$

where

$$
\boldsymbol{V}_{m,l} = \boldsymbol{\omega}_{m,l} + \boldsymbol{B}_{m,l}.
$$

Clearly, the energy equality (4.8) holds true with $\boldsymbol{u}_{m,l}$ in place of $\boldsymbol{u}_m$ and $\boldsymbol{B}_{m,l}$ in place of $\boldsymbol{B}_m$ respectively. In particular, both $\{\boldsymbol{u}_{m,l}\}$ and $\{\boldsymbol{B}_{m,l}\}$ are bounded in $V^2(Q)$. Thus, by a standard reflexivity argument along with Banach-Alaoglu's compactness lemma, eventually passing to a subsequence, we may assume there exist $\boldsymbol{u}_m \in V^2_{\mathrm{div}}(Q)$ and $\boldsymbol{B}_m \in V^2_{\mathrm{div}}(Q)$ such that

$$
\nabla \boldsymbol{u}_{m,l} \to \nabla \boldsymbol{u}_m, \quad \nabla \boldsymbol{B}_{m,l} \to \nabla \boldsymbol{B}_m \quad \textit{weakly in} \quad L^2(Q), \tag{4.13}
$$

$$
\boldsymbol{u}_{m,l} \to \boldsymbol{u}_m, \quad \boldsymbol{B}_{m,l} \to \boldsymbol{B}_m \quad \textit{weakly* in} \quad L^\infty(0,T;L^2) \quad \text{as} \quad l \to +\infty. \tag{4.14}
$$

Furthermore, by Lions-Aubin's compactness lemma we see that

$$
\boldsymbol{u}_{m,l} \to \boldsymbol{u}_m, \quad \boldsymbol{B}_{m,l} \to \boldsymbol{B}_m \quad \textit{strongly in} \quad L^2(Q) \quad \text{as} \quad l \to +\infty. \tag{4.15}
$$

Hence, thanks to (4.13), (4.14) and (4.15) we are in a position to carry out the passage to the limit $l \to +\infty$ in the weak formulation of (4.9)–(4.11). Accordingly, there exists

$p_m \in L^2(0,T;L^2_{\mathrm{loc}})$ such that $(\boldsymbol{u}_m, p_m, \boldsymbol{B}_m)$ is a weak solution to (4.1)–(4.4). Verifying that $\boldsymbol{u}_m$ and $\boldsymbol{B}_m$ satisfying the energy equality (4.8), it follows that

$$\nabla \boldsymbol{u}_{m,l} \to \nabla \boldsymbol{u}_m, \quad \nabla \boldsymbol{B}_{m,l} \to \nabla \boldsymbol{B}_m \quad \textit{strongly in} \quad L^2(Q) \quad \text{as} \quad l \to +\infty. \tag{4.16}$$

As $V^2(Q) \hookrightarrow L^4(Q)$ from (4.16) we infer

$$\boldsymbol{u}_{m,l} \to \boldsymbol{u}_m, \quad \boldsymbol{B}_{m,l} \to \boldsymbol{B}_m \quad \textit{strongly in} \quad L^4(Q) \quad \text{as} \quad l \to +\infty. \tag{4.17}$$

Next, applying $\nabla\times$ to both sides of (4.9) and combining the result with (4.10), we are led to

$$\partial_t \boldsymbol{V}_{m,l} - \Delta \boldsymbol{V}_{m,l} = -\nabla \times \Big(\frac{\boldsymbol{V}_{m,l}}{1 + \delta_m |\boldsymbol{B}_{m,l}| + \beta_l |\boldsymbol{V}_{m,l}|} \times \boldsymbol{u}_{m,l} \Big) + \nabla \times \boldsymbol{h} \quad \text{in} \quad Q, \tag{4.18}$$

where $\boldsymbol{h} = \boldsymbol{g} + \boldsymbol{f}$. By using a routine smoothing argument one gets $\boldsymbol{V}_{m,l} \in V^2(Q_r)$ for all $\overline{Q}_r \subset Q$.

Now, let $\overline{Q}_r = \overline{Q_r(X_0)} \subset Q$ be arbitrarily chosen. Let $\theta \in C^\infty_{\mathrm{c}}(B_r \times (t_0 - r^2, t_0])$ be a test function suitable for $Q_{r/2}$. Testing (4.18) by $\theta^2 \boldsymbol{V}_{m,l}$, we get

$$\begin{aligned}
&\frac{1}{2} \int\limits_{B_r} \theta^2(t) |\boldsymbol{V}_{m,l}(t)|^2 dx + \int\limits_{t_0 - r^2}^{t} \int\limits_{B_r} \theta^2 |\nabla \boldsymbol{V}_{m,l}|^2 dx ds \\
&\quad = \frac{1}{2} \int\limits_{t_0 - r^2}^{t} \int\limits_{B_r} (\partial_t \theta^2 + \Delta \theta^2) |\boldsymbol{V}_{m,l}|^2 dx ds \\
&\qquad - \int\limits_{t_0 - r^2}^{t} \int\limits_{B_r} \Big(\frac{\boldsymbol{V}_{m,l}}{1 + \delta_m |\boldsymbol{B}_{m,l}| + \beta_l |\boldsymbol{V}_{m,l}|} \times \boldsymbol{u}_{m,l} - \boldsymbol{h} \Big) \cdot \nabla \times (\theta^2 \boldsymbol{V}_{m,l}) dx ds
\end{aligned} \tag{4.19}$$

for a. e. $t \in (t_0 - r^2, t_0)$. From the above identity using the embedding $V^2(Q_r) \hookrightarrow L^4(Q_r)$, it is readily seen that

$$\begin{aligned}
&\Big(\int\limits_{Q_r} \theta^4 |\boldsymbol{V}_{m,l}|^4 dx dt \Big)^{1/2} \\
&\le c \operatorname*{ess\,sup}_{t \in (t_0 - r^2, t_0)} \int\limits_{B_r} \theta^2(t) |\boldsymbol{V}_{m,l}(t)|^2 dx + c \int\limits_{Q_r} \theta^2 |\nabla \boldsymbol{V}_{m,l}|^2 + r^{-2} |\boldsymbol{V}_{m,l}|^2 + |\boldsymbol{h}|^2 dx dt \\
&\le c r^{-2} (1 + \|\boldsymbol{u}_{m,l}\|_4^2) \int\limits_{Q_r} |\boldsymbol{V}_{m,l}|^2 dx dt + c \|\boldsymbol{h}\|_2^2 \\
&\qquad + \hat{C} \|\boldsymbol{u}_{m,l}\|_{4,Q_r} \Big(\int\limits_{Q_r} \theta^4 |\boldsymbol{V}_{m,l}|^4 dx dt \Big)^{1/2},
\end{aligned} \tag{4.20}$$

with an absolute constant $\hat{C} > 0$. As $\boldsymbol{u}_m \in L^4(Q)$, we may choose $0 < r < \sqrt{t_0}$ such that $\hat{C}\|\boldsymbol{u}_m\|_{4,Q_r} \leq \frac{1}{4}$. Observing (4.17), there exists $l_0 \in \mathbb{N}$ such that $\hat{C}\|\boldsymbol{u}_{m,l}\|_{4,Q_r} \leq \frac{1}{2}$ for all $l \geq l_0$. Accordingly, (4.20) implies

$$(4.21) \qquad \left(\int\limits_{Q_r} \theta^4 |\boldsymbol{V}_{m,l}|^4 dxdt \right)^{1/2} \leq cr^{-2}(1 + \|\boldsymbol{u}_{m,l}\|_4^2) \int\limits_{Q_r} |\boldsymbol{\omega}_{m,l} + \boldsymbol{V}_{m,l}|^2 dxdt + c\|\boldsymbol{h}\|_2^2$$

for $l \geq l_0$. Since the right of (4.21) is bounded independently of $l \in \mathbb{N}$, by a constant $C(\boldsymbol{u}_0, \boldsymbol{B}_0, \boldsymbol{f}, \boldsymbol{g})$ by virtue of the lower semi continuity of the norm from (4.21) together with (4.19) and (4.20) we get

$$(4.22) \qquad \|\nabla \boldsymbol{V}_m\|_{2,Q_{r/2}} + \|\boldsymbol{V}_m\|_{L^\infty(t_0 - r^2/4, t_0; L^2(B_{r/2}))} + \|\boldsymbol{V}_m\|_{4,Q_{r/2}} \leq C(\boldsymbol{u}_0, \boldsymbol{B}_0, \boldsymbol{f}, \boldsymbol{g}),$$

where $\boldsymbol{V}_m = \boldsymbol{B}_m + \nabla \times \boldsymbol{u}_m$. By applying a standard covering argument, since $\boldsymbol{B}_m \in L^2$ we see that $\nabla \times \boldsymbol{u}_m \in V^2(Q_r)$ for all $\overline{Q}_r \subset Q$. Whence, the assertion follows from the inequality

$$\|\nabla \boldsymbol{u}_m\|_{2,Q_{r/2}} \leq cr^{-1}(\|\boldsymbol{u}_m\|_{2,Q_r} + \|\nabla \times \boldsymbol{u}_m\|_{2,Q_r})$$

which completes the proof of the lemma. ∎

Next, we are going to carry out the passage to the limit $m \to +\infty$, which can be done by an analogous argument used in the proof of Lemma 4.1. Observing the energy equality (4.8), we find that both $\{\boldsymbol{u}_m\}$ and $\{\boldsymbol{B}_m\}$ are bounded in $V^2(Q)$. Eventually passing to a subsequence, we get the existence of $\boldsymbol{u}, \boldsymbol{B} \in V^2_{\mathrm{div}}(Q)$ such that

$$(4.23) \qquad \nabla \boldsymbol{u}_m \to \nabla \boldsymbol{u}, \quad \nabla \boldsymbol{B}_m \to \nabla \boldsymbol{B} \quad \textit{weakly in} \quad L^2(Q),$$

$$(4.24) \qquad \boldsymbol{u}_m \to \boldsymbol{u}, \quad \boldsymbol{B}_{m,l} \to \boldsymbol{B}_m \quad \textit{weakly}^* \textit{ in} \quad L^\infty(0,T;L^2) \quad \text{as} \quad m \to +\infty.$$

Furthermore, by Lions-Aubin's compactness lemma we see that

$$(4.25) \qquad \boldsymbol{u}_m \to \boldsymbol{u}, \quad \boldsymbol{B}_m \to \boldsymbol{B} \quad \textit{strongly in} \quad L^2(Q) \quad \text{as} \quad m \to +\infty.$$

With the aid of (4.23), (4.24) and (4.25) we are in a position to carry out the passage to the limit $m \to +\infty$ in the weak formulation of (4.1)–(4.4), which yields a weak solution $(\boldsymbol{u}, p, \boldsymbol{B})$ to (1.1)–(1.4).

Our next aim is to get a strong L^4 convergence of $\boldsymbol{u}_m$.

Lemma 4.2. *Let $\{(\boldsymbol{u}_m, p_m, \boldsymbol{B}_m)\}$ be a sequence of weak solutions to* (4.1)–(4.4) *obtained by Lemma 4.1. Furthermore, suppose* (4.23)–(4.25)*. Then, for every $\overline{Q}_r \subset Q$ there holds*

$$(4.26) \qquad \boldsymbol{u}_m \to \boldsymbol{u} \quad \textit{strongly in} \quad \boldsymbol{L}^4(Q_r) \quad \textit{as} \quad m \to +\infty.$$

In addition, for every $X_0 \in Q$ there exists $0 < r = r(X_0) < \sqrt{t_0}$ such that

$$(4.27) \qquad \begin{aligned} &\|\nabla \boldsymbol{\omega}_m\|_{2,Q_r} + \|\boldsymbol{\omega}_m\|_{L^\infty(t_0 - r^2, t_0; L^2(B_r))} + \|\boldsymbol{\omega}_m\|_{4,Q_r} \\ &\qquad \leq C(\boldsymbol{u}_0, \boldsymbol{B}_0, \boldsymbol{f}, \boldsymbol{g}) \quad \forall\, m \in \mathbb{N}. \end{aligned}$$

Proof Let $m \in \mathbb{N}$. In view of Lemma 4.1, taking the sum of (4.1) and (4.2), we see that $\boldsymbol{V}_m = \boldsymbol{\omega}_m + \boldsymbol{B}_m \in V^2_{\rm loc}(Q)$ is a weak solution to the following system

$$\partial_t \boldsymbol{V}_m - \Delta \boldsymbol{V}_m = -\nabla \times \Big(\frac{\boldsymbol{V}_m}{1+\delta_m |\boldsymbol{B}_m|} \times \boldsymbol{u}_m \Big) + \nabla \times \boldsymbol{h} \quad \text{in} \quad Q. \tag{4.28}$$

Here $V^2_{\rm loc}(Q)$ contains all $\boldsymbol{\varphi} \in L^2(Q)$ such that $\varphi|_{Q_r} \in V^2(Q_r)$ for all $\overline{Q}_r \subset Q$.
Clearly, there exists $\boldsymbol{v}_m \in V^2_{\rm loc}(Q)$ such that $\nabla \times \boldsymbol{v}_m = \boldsymbol{V}_m$. Thus, from (4.28) we infer that

$$\partial_t \boldsymbol{v}_m - \Delta \boldsymbol{v}_m = -\nabla \pi_m - \frac{\boldsymbol{V}_m}{1+\delta_m |\boldsymbol{B}_m|} \times (\boldsymbol{v}_m - \boldsymbol{b}_m) + \boldsymbol{h} \quad \text{in} \quad Q, \tag{4.29}$$

where $\boldsymbol{b}_m = \boldsymbol{v}_m - \boldsymbol{u}_m$. By the definition of $\boldsymbol{v}_m$ we have $\nabla \times \boldsymbol{b}_m = \boldsymbol{B}_m$.

Let $\overline{Q}_r \subset Q$ be fixed. Eventually, replacing $\boldsymbol{v}_m$ by $\boldsymbol{v}_m(t) - (\boldsymbol{v}_m(t))_{x_0,r}$ $(t \in t_0 - r^2, t_0)$, observing (4.23), (4.24) and (4.25), by virtue of Sobolev's embedding theorem we easily verify that

$$\boldsymbol{V}_m \to \boldsymbol{V} \quad \textit{weakly in} \quad L^2(Q_r), \tag{4.30}$$

$$\boldsymbol{b}_m \to \boldsymbol{b} \quad \textit{strongly in} \quad L^6(Q_r) \quad \text{as} \quad m \to +\infty. \tag{4.31}$$

Indeed, we note that $|\boldsymbol{b}_m(t)_{x_0,B_r}| = |\boldsymbol{u}_m(t)_{x_0,B_r}| \le \|\boldsymbol{u}_m\|_{L^\infty(0,T;L^2)}$. Consequently, by Sobolev-Poincaré's inequality we see that $\|\boldsymbol{b}_m\|_{q,Q_r} \le c\|\boldsymbol{u}_m\|_{L^\infty(0,T;L^2)} + c\|\boldsymbol{B}_m\|_{L^\infty(0,T;L^2)}$ for every $1 \le q < +\infty$. Once more appealing to (4.25), eventually passing to a subsequence we may assume that

$$\boldsymbol{B}_m \to \boldsymbol{B} \quad \textit{a.e. in} \quad Q \quad \text{as} \quad m \to +\infty. \tag{4.32}$$

By means of Vitali's convergence theorem, making use of (4.31) and (4.32), we get

$$\frac{\boldsymbol{b}_m}{1+\delta_m |\boldsymbol{B}_m|} \to \boldsymbol{b} \quad \textit{strongly in} \quad L^6(Q_r) \quad \text{as} \quad m \to +\infty. \tag{4.33}$$

Next, we define the local pressure

$$\begin{aligned} \nabla \pi_{m,1} &= \boldsymbol{E}_{B_r}(\Delta \boldsymbol{v}_m), \\ \nabla \pi_{m,2} &= \boldsymbol{E}_{B_r}\Big(- \frac{\boldsymbol{V}_m}{1+\delta_m |\boldsymbol{B}_m|} \times (\boldsymbol{v}_m - \boldsymbol{b}_m) + \boldsymbol{h} \Big), \\ \nabla \pi_{m,\rm hm} &= -\boldsymbol{E}_{B_r}(\boldsymbol{v}_m), \end{aligned}$$

where $\boldsymbol{E}_{B_r} : W^{-1,q}(B_r) \to W^{-1,q}(B_r)$ stands for the projection defined by the Stokes equation. Note that the restriction of $\boldsymbol{E}_{B_r}$ to $L^q(Q_r)$ $(1 < q < +\infty)$ defines a projection in $L^q(Q_r)$ (cf. [23, 24] for details). We also note that $\pi_{m,\rm hm}(t)$ is harmonic in B_r for a. e. all $t \in (t_0 - r^2, t_0)$. As it has been proved in [23],(4.29) implies that the function $\boldsymbol{z}_m = \boldsymbol{v}_m + \nabla \pi_{m,\rm hm} \in V^2(Q_r)$ solves the following system in sense of distributions

$$\partial_t \boldsymbol{z}_m - \Delta \boldsymbol{z}_m = -\nabla(\pi_{m,1} + \pi_{m,2}) - \frac{\boldsymbol{V}_m}{1+\delta_m |\boldsymbol{B}_m|} \times \boldsymbol{u}_m + \boldsymbol{h} \quad \text{in} \quad Q_r, \tag{4.34}$$

Let $\phi \in C_{\mathrm{c}}^{\infty}(Q_r)$ be a non-negative cut-off function. Testing (4.34) by $\phi \boldsymbol{z}_m$, we obtain the following energy equality

$$\begin{aligned}
&\int\limits_{Q_r} \phi |\nabla \boldsymbol{z}_m|^2 dxdt \\
&\qquad = \frac{1}{2} \int\limits_{Q_r} (\partial_t \phi + \Delta \phi) |\boldsymbol{z}_m|^2 dxdt + \int\limits_{Q_r} \Big(\frac{\boldsymbol{V}_m}{1 + \delta_m |\boldsymbol{B}_m|} \times \boldsymbol{b}_m + \boldsymbol{h} \Big) \cdot \phi \boldsymbol{z}_m dxdt \\
&\qquad\qquad + \int\limits_{Q_r} (\pi_{m,1} + \pi_{m,2}) \nabla \phi \cdot \boldsymbol{z}_m dxdt. \qquad (4.35)
\end{aligned}$$

Verifying

$$\Big\| \frac{\boldsymbol{V}_m}{1 + \delta_m |\boldsymbol{B}_m|} \times \boldsymbol{u}_m \Big\|_{L^{3/2}(0,T;L^{6/5})} \le \|\boldsymbol{V}_m\|_2 \|\boldsymbol{u}_m\|_{L^6(0,T;L^3)} \le C(\boldsymbol{u}_0, \ldots),$$

we may estimate the pressure $\pi_{m,2}$ in $L^{3/2}(Q_r)$ by using the Sobolev-Poincaré inequality as follows

$$\begin{aligned}
&\|\pi_{m,2}\|_{3/2, Q_r} \le c \|\nabla \pi_{m,2}\|_{L^{3/2}(t_0 - r^2, t_0; L^{6/5}(B_r))} \\
&\qquad \le c \Big\| \frac{\boldsymbol{V}_m}{1 + \delta_m |\boldsymbol{B}_m|} \times \boldsymbol{u}_m \Big\|_{L^{3/2}(0,T;L^{6/5})} + c \|\boldsymbol{h}\|_2 \le C(\boldsymbol{u}_0, \ldots). \qquad (4.36)
\end{aligned}$$

Furthermore, we immediately get

$$\|\pi_{m,1}\|_{2, Q_r} \le c \|\nabla \boldsymbol{v}_m\|_2 \le c \|\nabla \boldsymbol{u}_m\|_2 + c \|\boldsymbol{B}_m\|_2 \le C(\boldsymbol{u}_0, \ldots). \qquad (4.37)$$

Observing (4.25) along with (4.31), we find

$$\boldsymbol{v}_m \to \boldsymbol{v} \quad \textit{strongly in} \quad L^3(Q_r) \quad \text{as} \quad m \to +\infty, \qquad (4.38)$$

where $\boldsymbol{v} = \boldsymbol{u} + \boldsymbol{b}$. Thus, having

$$\nabla \pi_{m,\mathrm{hm}} \to \nabla \pi_{\mathrm{hm}} \quad \text{strongly in} \quad L^3(Q_r) \quad \text{as} \quad m \to \infty, \qquad (4.39)$$

where $\nabla \pi_{\mathrm{hm}} = -\boldsymbol{E}_{B_r}(\boldsymbol{v})$, it follows that

$$\boldsymbol{z}_m \to \boldsymbol{z} \quad \textit{strongly in} \quad L^3(Q_r) \quad \text{as} \quad m \to +\infty. \qquad (4.40)$$

Now, with help of (4.36), (4.37) and (4.40) we get

$$\lim_{m \to \infty} \int\limits_{Q_r} (\pi_{m,1} + \pi_{m,2}) \nabla \phi \cdot \boldsymbol{z}_m dxdt = \int\limits_{Q_r} (\pi_1 + \pi_2) \nabla \phi \cdot \boldsymbol{z} dxdt,$$

where

$$\begin{aligned}
&\nabla \pi_1 = \boldsymbol{E}_{B_r}(\Delta \boldsymbol{v}), \\
&\nabla \pi_2 = \boldsymbol{E}_{B_r}(-\boldsymbol{V} \times \boldsymbol{u} + \boldsymbol{h}).
\end{aligned}$$

On the other hand, making use of (4.33), together with (4.25) and (4.40) we see that

$$\lim_{m\to\infty}\int\limits_{Q_r}\Big(\frac{\boldsymbol{V}_m}{1+\delta_m|\boldsymbol{B}_m|}\times \boldsymbol{b}_m+\boldsymbol{h}\Big)\cdot\phi\boldsymbol{z}_m dxdt=\int\limits_{Q_r}(\boldsymbol{V}\times\boldsymbol{b}+\boldsymbol{h})\cdot\phi\boldsymbol{z}dxdt.$$

Furthermore, thanks to (4.40) we obtain

$$\lim_{m\to\infty}\frac{1}{2}\int\limits_{Q_r}(\partial_t\phi+\Delta\phi)|\boldsymbol{z}_m|^2dxdt=\frac{1}{2}\int\limits_{Q_r}(\partial_t\phi+\Delta\phi)|\boldsymbol{z}|^2dxdt.$$

Hence, we are in the position to carry out the passage to the limit $m\to+\infty$ in (4.35) to get

$$\begin{aligned}&\lim_{m\to\infty}\int\limits_{Q_r}\phi|\nabla\boldsymbol{z}_m|^2dxdt\\ &\quad=\frac{1}{2}\int\limits_{Q_r}(\partial_t\phi+\Delta\phi)|\boldsymbol{z}|^2dxdt+\int\limits_{Q_r}(\boldsymbol{V}\times\boldsymbol{b}+\boldsymbol{h})\cdot\phi\boldsymbol{z}dxdt+\int\limits_{Q_r}(\pi_1+\pi_2)\nabla\phi\cdot\boldsymbol{z}dxdt.\end{aligned}\tag{4.41}$$

Accordingly, we see that $\boldsymbol{z}\in V^2(Q_r)$ and

$$\partial_t\boldsymbol{z}-\Delta\boldsymbol{z}=-\nabla(\pi_1+\pi_2)-\boldsymbol{V}\times\boldsymbol{u}+\boldsymbol{h}\quad\text{in}\quad Q_r,\tag{4.42}$$

in sense of distributions. Taking into account that $\boldsymbol{z}\in L^4(Q_r)$, and $\boldsymbol{V}\times\boldsymbol{u}\in L^{4/3}(Q)$, we obtain the following energy equality

$$\begin{aligned}&\int\limits_{Q_r}\phi|\nabla\boldsymbol{z}|^2dxdt\\ &\quad=\frac{1}{2}\int\limits_{Q_r}(\partial_t\phi+\Delta\phi)|\boldsymbol{z}|^2dxdt+\int\limits_{Q_r}(\boldsymbol{V}\times\boldsymbol{b}+\boldsymbol{h})\cdot\phi\boldsymbol{z}dxdt\\ &\qquad\quad+\int\limits_{Q_r}(\pi_1+\pi_2)\nabla\phi\cdot\boldsymbol{z}dxdt.\end{aligned}\tag{4.43}$$

Thus, observing (4.25), combining (4.41) and (4.43) using a well-known liminf-limsup argument noticing that $\sqrt{\phi}\nabla\boldsymbol{z}_m\to\sqrt{\phi}\nabla\boldsymbol{z}$ weakly in $L^2(Q_r)$ we get

$$\sqrt{\phi}\nabla\boldsymbol{z}_m\to\sqrt{\phi}\nabla\boldsymbol{z}\quad\text{strongly in}\quad L^2(Q_r)\quad\text{as}\quad m\to\infty.$$

On the other hand, since π_{hm} is harmonic, thanks to (4.39) we get

$$\sqrt{\phi}\nabla^2\pi_{m,\mathrm{hm}}\to\sqrt{\phi}\nabla^2\pi_{\mathrm{hm}}\quad\text{strongly in}\quad L^2(Q_r)\quad\text{as}\quad m\to\infty.$$

As $\nabla\boldsymbol{v}_m=\nabla\boldsymbol{z}_m-\nabla^2\pi_{m,\mathrm{hm}}$ a. e. in Q_r, we arrive at

$$\sqrt{\phi}\nabla\boldsymbol{v}_m\to\sqrt{\phi}\nabla\boldsymbol{v}\quad\text{strongly in}\quad L^2(Q_r)\quad\text{as}\quad m\to\infty.$$

Hence, thanks to the embedding $V^2(Q_r) \hookrightarrow L^4(Q_r)$ along with (4.31) we get

$$\sqrt{\phi}\boldsymbol{u}_m \to \sqrt{\phi}\boldsymbol{u} \quad \text{strongly in} \quad L^4(Q_r) \qquad \text{as} \quad m \to \infty.$$

Since the above statement holds for any cylinder $Q_r \subset Q$, we get the first claim (4.26) of the lemma.

Now, it remains to verify (4.27). In fact, according to Lemma 4.1, we have $\boldsymbol{V}_m \in L^4(Q_r)$, which implies that $\frac{\boldsymbol{V}_m}{1+\delta_m|\boldsymbol{B}_m|} \times \boldsymbol{u}_m \in L^2(Q_r)$. This allows us to test (4.28) with $\theta^2\boldsymbol{V}_m$, where $\theta \in C_c^\infty(B_r \times (t_0 - r^2, t_0])$. Arguing as in the proof of Lemma 4.1, we obtain

$$\begin{aligned}
&\frac{1}{2}\int\limits_{B_r} \theta^2(t)|\boldsymbol{V}_m(t)|^2 dx + \int\limits_{t_0-r^2}^{t} \int\limits_{B_r} \theta^2|\nabla \boldsymbol{V}_m|^2 dxds \\
&\quad = \frac{1}{2}\int\limits_{t_0-r^2}^{t} \int\limits_{B_r} (\partial_t\theta^2 + \Delta\theta^2)|\boldsymbol{V}_m|^2 dxds \\
&\qquad - \int\limits_{t_0-r^2}^{t} \int\limits_{B_r} \Big(\frac{\boldsymbol{V}_m}{1+\delta_m|\boldsymbol{B}_m|} \times \boldsymbol{u}_m - \boldsymbol{h}\Big) \cdot \nabla \times (\theta^2\boldsymbol{V}_m) dxds
\end{aligned}$$

for a. e. $t \in (t_0 - r^2, t_0)$, which leads to

$$\begin{aligned}
&\Big(\int\limits_{Q_r} \theta^4|\boldsymbol{V}_m|^4 dxdt\Big)^{1/2} \\
&\quad \le cr^{-2}(1 + \|\boldsymbol{u}_m\|_4^2)C(\boldsymbol{u}_0, \ldots) + \hat{C}\|\boldsymbol{u}_m\|_{4,Q_r}\Big(\int\limits_{Q_r} \theta^4|\boldsymbol{V}_m|^4 dxdt\Big)^{1/2}. \qquad (4.44)
\end{aligned}$$

Whence, the proof of (4.27) can be completed by a similar argument to the proof of Lemma 4.1, by using the strong L^4 convergence (4.26). ■

5 Proof of Theorem 1.2

Let $(\boldsymbol{u}_m, p_m, \boldsymbol{B}_m) \in V^2_{\mathrm{div}}(Q) \times L^2(0, T; L^2_{\mathrm{loc}}) \times V^2_{\mathrm{div}}(Q)$ be a weak solution to the approximate system (4.1)–(4.4) such that $\nabla\boldsymbol{u}_m \in V^2_{\mathrm{loc}}(Q)$ ($m \in \mathbb{N}$), which can be guaranteed by Lemma 4.1 (for the definition of $V^2_{\mathrm{loc}}(Q)$ see Section 4).

In our discussion below we use the following notation. Let $X_0 = (x_0, t_0) \in Q$.

$$\begin{aligned}
E_m(r) = E_m(r, X_0) &:= \Big(\fint_{Q_r(X_0)} |\boldsymbol{B}_m - (\boldsymbol{B}_m)_{r,X_0}|^4 dxdt\Big)^{1/4}, \\
F_m(r) = F_m(r, X_0) &:= \Big(r^{-2} \int\limits_{Q_r(X_0)} |\nabla\boldsymbol{B}_m|^2 dxdt\Big)^{1/2}, \\
H_m(r) = H_m(r, X_0) &:= \|\boldsymbol{u}_m\|_{4,Q_r} + r^{-1}\|\boldsymbol{g}\|_{2,Q_r} \quad 0 < r < \sqrt{t_0}.
\end{aligned}$$

Next, we define the set of possible singularities of B by means of $\Sigma(\boldsymbol{B}) = \cup_{k=1}^{\infty}\Sigma_k \cup \Sigma_\infty$, where

$$\Sigma_k := \bigcup_{0<\rho<T}\bigcap_{0<r\leq\rho}\left\{X_0 \in \mathbb{R}^2 \times (r,T) \,\middle|\, \liminf_{m\to\infty} F_m(r,X_0) \geq \frac{1}{k}\right\}, \quad k \in \mathbb{N},$$
$$\Sigma_\infty := \left\{X_0 \in Q \,\middle|\, \sup_{0<r<\sqrt{t_0}} |\boldsymbol{B}_{r,X_0}| = +\infty\right\}.$$

Let $Q_r = Q_r(X_0) \subset Q$ be any cylinder such that condition (2.13) is fulfilled for $\boldsymbol{B} = \boldsymbol{B}_m$ and $\boldsymbol{u} = \boldsymbol{u}_m$, i. e.

$$C_1\left\{F_m(r)^2 + \|\boldsymbol{u}_m\|^2_{4,Q_r}\right\} \leq \frac{1}{2}. \tag{5.1}$$

As stated in Remark 2.4, the condition (2.13) implies (2.15). Thus, (5.1) implies

$$E_m(r/2) \leq C_2(1 + |(\boldsymbol{B}_m)_{r,X_0}|^2)\left\{F_m(r) + F_m(r)^2 + H_m(r) + H_m(r)^2\right\}. \tag{5.2}$$

On the other hand, (2.16) with $\boldsymbol{B} = \boldsymbol{B}_m$ and $\boldsymbol{u} = \boldsymbol{u}_m$ reads

$$F_m(r/2) \leq C_3(1 + |(\boldsymbol{B}_m)_{r,X_0}|)\left\{E_m(r) + E_m(r)^2 + H_m(r) + H_m(r)^2\right\}. \tag{5.3}$$

Let $X_0 \in Q \setminus \Sigma(\boldsymbol{B})$ be fixed. Set $d_0 = \sqrt{t_0}/2$ and $K = \overline{Q_{d_0}}$. Appealing to Lemma 4.2, and applying Sobolev's embedding theorem, we see that

$$\|\boldsymbol{u}_m\|_{8/(4-\lambda),K} \leq L \quad \forall\, m \in \mathbb{N}, \tag{5.4}$$

where $L = \text{const} > 0$ depends on $d_0, \boldsymbol{u}_0, \boldsymbol{B}_0, \boldsymbol{f}$ and $\boldsymbol{g}$ only. Furthermore, we may choose $0 < R_1 < d_0$ such that

$$C_1\|\boldsymbol{u}_m\|^2_{4,Q_{R_1}} \leq \frac{1}{16} \quad \forall\, m \in \mathbb{N}, \tag{5.5}$$

where C_1 stands for the constant appearing in (5.1). Using Hölder's inequality, recalling the assumption on $\boldsymbol{g}$ along with (5.4), it follows that

$$\begin{aligned} H_m(r,X_0) &\leq \left(\pi^{\lambda/8-1/4}\|\boldsymbol{u}_m\|_{8/(4-\lambda),K} + \|\boldsymbol{g}\|_{\mathcal{M}^{2,\lambda}(K)}\right) r^{(\lambda-2)/2} \\ &\leq C_4 r^{(\lambda-2)/2} \quad \forall \quad 0 < r \leq R_1. \end{aligned} \tag{5.6}$$

Next, we set

$$M := 512 \sup_{0<r<d(X_0/2)} (|\boldsymbol{B}|)_{r,X_0} + 1 < +\infty.$$

Let $0 < \alpha < \frac{2-\lambda}{2}$. We take $\tau > 0$ such that

$$2\tau^{1-\alpha}C_0(1+M^5) \leq \frac{1}{2} \quad \text{and} \quad \tau^\alpha \leq \frac{1}{2} \tag{5.7}$$

(Recall, the constant $C_0 > 0$ has been defined in Lemma 3.1).

Now, let $\varepsilon_0 = \varepsilon_0(\tau, M, L, K, \alpha), R_0 = R_0(\tau, M, L, K, \alpha)$ and $\delta_0 = \delta_0(\tau, M, L, K, \alpha)$ denote the numbers according to Lemma 3.2. In addition, we define $\varepsilon_1 > 0$ by the relation

$$2\tau^{-4}\varepsilon_1 = 1. \tag{5.8}$$

Next we may choose $0 < R_2 \le \min\{R_0, R_1\}$ such that the following conditions hold

$$C_2(1+M^2)(C_4 + C_4^2)R_2^{(\lambda-2)/2} \le \frac{1}{8}\min\{\varepsilon_0, \varepsilon_1\}, \tag{5.9}$$

$$2R_2^{\alpha} \le \frac{1}{2}\min\{\varepsilon_0, \varepsilon_1\}. \tag{5.10}$$

Now, we take $k \in \mathbb{N}$ such that

$$C_2(1+M^2)\Big\{\frac{1}{k} + \frac{1}{k^2}\Big\} \le \frac{1}{8}\min\{\varepsilon_0, \varepsilon_1\} \quad \text{and} \quad \frac{C_1}{k} \le \frac{1}{4}. \tag{5.11}$$

Owing to $X_0 \in Q \setminus \Sigma_k$ eventually replacing R_2 by a smaller number we may also assume that $\liminf_{m\to\infty} F_m(R_2, X_0) < \frac{1}{k}$. Accordingly we are able to select a subsequence $\{m_j\}$ such that

$$F_{m_j}(R_2, X_0) < \frac{1}{k} \quad \forall\, j \in \mathbb{N}. \tag{5.12}$$

Since $\boldsymbol{B}_m \to \boldsymbol{B}$ in $L^1(Q_{R_2})$ and $\delta_m \to 0$ as $m \to +\infty$, there exists $m_0 \in \mathbb{N}$ with the property

$$(|\boldsymbol{B}_m|)_{R_2,X_0} \le (|\boldsymbol{B}|)_{R_2,X_0} + \frac{1}{512} \le \frac{M}{512} \quad \text{and} \quad \delta_m \le \delta_0 \quad \forall\, m \ge m_0. \tag{5.13}$$

Observing (5.12), (5.11) and (5.5), we have

$$C_1\Big\{F_{m_j}(R_2, X_0) + 4\|\boldsymbol{u}_{m_j}\|_{4,Q_{R_2}(X_0)}\Big\} \le \frac{1}{2} \quad \forall\, j \in \mathbb{N}. \tag{5.14}$$

As (5.14) implies (5.2), employing (5.13), (5.12) and (5.6), we get

$$E_{m_j}(R_2/2, X_0) \le C_2(1+M^2)\Big\{\frac{1}{k} + \frac{1}{k^2} + (C_4 + C_4^2)R_2^{(\lambda-2)/2}\Big\} \tag{5.15}$$

for all $m_j \ge m_0$. In view of (5.11) and (5.9), (5.15) gives

$$E_{m_j}(R_2, X_0) \le \frac{1}{4}\min\{\varepsilon_0, \varepsilon_1\} \quad \forall\, m_j \ge m_0. \tag{5.16}$$

Set $R_3 = R_2/2$. Let $Y \in Q_{R_2}(X_0)$. Clearly,

$$E_{m_j}(R_3, Y) \le 2E_{m_j}(R_2, X_0) \le \frac{1}{2}\min\{\varepsilon_0, \varepsilon_1\}, \tag{5.17}$$

$$|(\boldsymbol{B}_{m_j})_{R_3,Y}| \le 256 \fint_{Q_{R_2}(X_0)} |\boldsymbol{B}_{m_j}|\,dxdt \le \frac{M}{2}. \tag{5.18}$$

We claim that for every $i \in \mathbb{N} \cup \{0\}$, there holds

$$E_{m_j}(\tau^i R_3, Y) \le 2^{-i}\tau^{\alpha i} E_{m_j}(R_3, Y) + (1-2^{-i})\tau^{\alpha i} R_3^\alpha, \tag{5.19}$$

$$|(\boldsymbol{B}_{m_j})_{\tau^i R_3, Y}| \le M - 2^{-i+1}. \tag{5.20}$$

In fact, for $i = 0$, (5.19) is trivially fulfilled, while (5.20) holds in view of (5.18).

Now, we assume that both (5.19) and (5.20) are fulfilled for $i \in N \cup \{0\}$. Then (5.19) together with (5.17) and (5.10) implies

$$E_{m_j}(\tau^i R_3, Y) + \tau^{\alpha i} R_3^\alpha \le \tau^{\alpha i}(E_{m_j}(R_3, Y) + 2R_3^\alpha) \le \tau^{\alpha i} \min\{\varepsilon_0, \varepsilon_1\}. \tag{5.21}$$

In particular, observing (5.20) we have

$$E_{m_j}(\tau^i R_3, Y) + (\tau^i R_3)^\alpha \le \varepsilon_0, \quad |(\boldsymbol{B}_{m_j})_{\tau^i R_3, Y}| \le M.$$

Thus, we are in a position to apply Lemma 3.2 with $R = \tau^i R_3$. This together with (5.19) gives

$$\begin{aligned} E_{m_j}(\tau^{i+1} R_3, Y) &\le 2\tau C_0(1+M^5)(E_{m_j}(\tau^i R_3, Y) + \tau^{\alpha i} R_3^\alpha) \\ &\le \frac{1}{2}\tau^\alpha E_{m_j}(\tau^i R_3, Y) + \frac{1}{2}\tau^{\alpha(i+1)} R_3^\alpha \\ &\le 2^{-(i+1)}\tau^{\alpha(i+1)} E_{m_j}(R_3, Y) + (1-2^{-(i+1)})\tau^{\alpha(i+1)} R_3^\alpha. \end{aligned} \tag{5.22}$$

Consequently (5.19) holds true for $i+1$.

Now, it remains to show (5.20) for $i+1$. First, from (5.19) along with (5.17) and (5.10) we infer

$$E_{m_j}(\tau^i R_3, Y) \le \tau^{\alpha i}(E_{m_j}(R_3, Y) + R_3^\alpha) \le \tau^{\alpha i}\varepsilon_1. \tag{5.23}$$

Using the triangle inequality and Jensen's inequality, we find

$$\begin{aligned} |(\boldsymbol{B}_{m_j})_{\tau^{i+1} R_3, Y}| &\le |(\boldsymbol{B}_{m_j})_{\tau^i R_3, Y}| + \left|(\boldsymbol{B}_{m_j})_{\tau^{i+1} R_3, Y} - (\boldsymbol{B}_{m_j})_{\tau^i R_3, Y}\right| \\ &\le |(\boldsymbol{B}_{m_j})_{\tau^i R_3, Y}| + 2\tau^{-4} E_{m_j}(\tau^i R_3, Y). \end{aligned}$$

Estimating the first term on the right by using (5.20) and the second one by the aid of (5.23) together with (5.7) and (5.8), we obtain

$$\begin{aligned} |(\boldsymbol{B}_{m_j})_{\tau^{i+1} R_3, Y}| &\le M - 2^{-i+1} + 2\tau^{-4}\tau^{\alpha i}\varepsilon_1 \\ &\le M - 2^{-i+1} + 2^{-i} = M - 2^{-i}. \end{aligned}$$

This completes the proof of (5.20) for $i+1$. Whence, the claim.

Since (5.19) holds true for every $Y \in \overline{Q_{R_3}(X_0)}$, by a standard iteration argument we get a constant $C_5 > 0$ such that

$$\left(\fint_{Q_r(Y)} |\boldsymbol{B}_{m_j} - (\boldsymbol{B}_{m_j})_{r,Y}|^4 dx \right)^{1/4} \le C_5 r^\alpha \quad \forall\, 0 < r < R_3, \quad \forall\, Y \in \overline{Q_{R_3}(X_0)}. \tag{5.24}$$

Thus, by means of the lower semi continuity of the L^4-norm the above inequality remains true for $\boldsymbol{B}$. Using the well-known integral characterization of the Hölder continuity in the parabolic setting[17], we obtain

$$\boldsymbol{B}|_{\overline{Q_{R_3}(X_0)}} \in C^{\alpha,\alpha/2}(\overline{Q_{R_3}(X_0)}) \tag{5.25}$$

(For the definition of $C^{\alpha,\alpha/2}(\overline{Q_{R_3}(X_0)})$ see appendix below). Clearly, (5.24) shows that

$$\lim_{r\to 0^+} E_{m_j}(r,Y) = 0 \quad \textit{uniformly for} \quad Y \in Q_{R_3}(X_0) \quad \text{and} \quad j \in \mathbb{N}.$$

Hence, in view of (5.3) we get $Y \notin \bigcup_{k=1}^{\infty} \Sigma_k$. Taking into account that $\boldsymbol{B}$ is Hölder continuous on $\overline{Q_{\rho_0}(X_0)}$, it follows that $Q_{\rho_0}(X_0) \subset Q \setminus \Sigma_\infty$ and thus

$$Q_{R_3}(X_0) \subset Q \setminus \Sigma(\boldsymbol{B}).$$

Consequently, $\Sigma(\boldsymbol{B})$ is a closed set. This completes the proof of the main theorem. ■

Theorem 5.1. *For the singular set constructed in the proof of Theorem 1.2 we have*

$$d\mathcal{P}_\beta(\Sigma(\boldsymbol{B})) = 0 \quad \forall\, \beta > 2, \tag{5.26}$$

where $d\mathcal{P}_\beta(\cdot)$ is the $\beta-$dimensional parabolic Hausdorff measure. In particular, the Hausdorff dimension of $\Sigma(\boldsymbol{B})$ satisfies $\dim_{\mathcal{H}}(\Sigma(\boldsymbol{B})) \le 2$.

Proof Let $2 < \beta \le \lambda$ be arbitrarily chosen. First we show that

$$d\mathcal{P}_\beta(\Sigma_k) = 0 \quad \forall\, k \in \mathbb{N}.$$

Let $X_0 \in \Sigma_k$. Fix $\varepsilon > 0$. Then there exists $0 < r(X_0) < \varepsilon$ and $m(X_0) \in \mathbb{N}$, such that

$$r(X_0)^{-2} \int\limits_{Q_{r(X_0)}(X_0)} |\nabla \boldsymbol{B}_m|^2 dxdt \ge \frac{1}{2k} \quad \forall\, m \ge m(X_0). \tag{5.27}$$

Clearly, the family of cylinders $\{Q_{r(X_0)}(X_0)\}_{X_0\in\Sigma_k}$ forms a covering of Σ_k. Thanks to the Vitali covering lemma there exists a pairwise disjoint family $\{Q_{r_i}(X_i)\}_{i\in\mathbb{N}}$ ($r_i := r(X_i)$) such that $\{Q_{3r_i}(X_i)\}_{i\in\mathbb{N}}$ covers Σ_k. Let $N \in \mathbb{N}$ be arbitrarily chosen. Set

$$m_N := \max\{m(X_1), \ldots, m(X_N)\}.$$

Then, from (5.27) with $X_0 = X_i$ $(i = 1, \ldots, N)$ and $m = m_N$ we infer

$$\begin{aligned}\sum_{i=1}^{N} r_i^\beta \le \varepsilon^{\beta-2} \sum_{i=1}^{N} r_i^2 \le 2\varepsilon^{\beta-2} k \sum_{i=1}^{N} \int\limits_{Q_{r_i}(X_i)} |\nabla \boldsymbol{B}_{m_N}|^2 dxdt \le 2\varepsilon^{\beta-2} k \int\limits_{Q} |\nabla \boldsymbol{B}_{m_N}|^2 \\ \le \varepsilon^{\beta-2} k C(\|\boldsymbol{u}_0\|_2, \ldots).\end{aligned}$$

This shows that

$$\sum_{i=1}^{\infty} r_i^\beta \le \varepsilon^{\beta-2} k C(\|\boldsymbol{u}_0\|_2, \ldots). \tag{5.28}$$

Consequently, $d\mathcal{P}_\beta(\Sigma_k) = 0$, which implies that $d\mathcal{P}_\beta\Big(\bigcup_{k=1}^\infty \Sigma_k\Big) = 0$.

Now, it remains to prove that $d\mathcal{P}_\beta(\Sigma_\infty) = 0$. As we will see below this follows easily from the following implication

$$\sup_{0<r<\sqrt{t_0}} r^{-\beta} \int\limits_{Q_r(X_0)} |\nabla B|^2 dxdt < +\infty \quad \Longrightarrow \quad X_0 \notin \Sigma_\infty, \quad X_0 \in Q. \tag{5.29}$$

Indeed, let $X_0 \in Q$ such that the condition on the left in (5.29) holds true. Choose $0 < \rho_0 < \sqrt{t_0}$ sufficiently small (specified below) and set $r_i = 2^{-i}\rho_0 \quad (i \in \mathbb{N})$.

Fix $i \in \mathbb{N}$. By using the parabolic Poincaré-type inequality (see Lemma A.1, appendix below), arguing as in the proof of (2.12), we estimate

$$\begin{aligned}
&\fint_{Q_{r_i}} |\boldsymbol{B} - \boldsymbol{B}_{r_i,X_0}|^2 dxdt \\
&\quad \le c(1 + |\boldsymbol{B}_{r_i,X_0}|^2) r_i^{-2} \int\limits_{Q_{r_i}} |\nabla \boldsymbol{B}|^2 dxdt \\
&\quad\quad + c(1 + |\boldsymbol{B}_{r_i,X_0}|^2) r^{-2} \int\limits_{Q_{r_i}} (|\boldsymbol{g}|^2 + |\boldsymbol{u}|^2) dxdt \\
&\quad\quad + C_6 \Bigg\{ r_i^{-2} \int\limits_{Q_{r_i}} |\nabla \boldsymbol{B}|^2 + \bigg(\int\limits_{Q_{r_i}} |\boldsymbol{u}|^4 dxdt \bigg)^{1/2} \Bigg\} \fint_{Q_{r_i}} |\boldsymbol{B} - \boldsymbol{B}_{r_i,X_0}|^2 dxdt
\end{aligned} \tag{5.30}$$

for an absolute constant $C_6 > 0$. Due to $\boldsymbol{u} \in L^4(Q)$ and our assumption on X_0 we may choose ρ_0 sufficiently small such that the numerical value in $\{\ldots\}$ is less than $\frac{1}{2C_6}$, which leads to

$$\begin{aligned}
&\fint_{Q_{r_i}} |\boldsymbol{B} - \boldsymbol{B}_{r_i,X_0}|^2 dxdt \\
&\quad \le 2c(1 + |\boldsymbol{B}_{r_i,X_0}|^2) r_i^{-2} \int\limits_{Q_{r_i}} |\nabla \boldsymbol{B}|^2 dxdt \\
&\quad\quad + 2c(1 + |\boldsymbol{B}_{r_i,X_0}|^2) r_i^{-2} \int\limits_{Q_{r_i}} (|\boldsymbol{g}|^2 + |\boldsymbol{u}|^2) dxdt.
\end{aligned} \tag{5.31}$$

Appealing to Lemma 4.2, we see that $\boldsymbol{u} \in L^q_{\rm loc}(Q)$ for all $1 \le q < +\infty$. In particular, $\boldsymbol{u} \in \mathcal{M}^{2,\lambda}(Q_{\sqrt{t_0}/2})$. Recalling that $\boldsymbol{g} \in \mathcal{M}^{2,\lambda}(Q)$ and $\beta \le \lambda$ from (5.31), we deduce that

$$\fint_{Q_{r_i}} |\boldsymbol{B} - \boldsymbol{B}_{r_i,X_0}|^2 dxdt \le c(1 + |\boldsymbol{B}_{r_i,X_0}|^2) r_i^{\beta-2} \tag{5.32}$$

with a constant $c > 0$ depending neither on r_i nor on ρ_0. Using the triangle inequality and employing (5.32), it follows that

$$\Big| |\boldsymbol{B}_{r_{i+1},X_0}| - |\boldsymbol{B}_{r_i,X_0}| \Big| \le C_7 (1 + |\boldsymbol{B}_{r_i,X_0}|) r_i^{(\beta-2)/2}, \tag{5.33}$$

where $C_7 = \text{const} > 0$ is independent on r_i and ρ_0. Thus, eventually replacing ρ_0 by a smaller one, we may assume that

$$C_7 \sum_{i=0}^{\infty} r_i^{(\beta-2)/2} = C_7 \rho_0^{(\beta-2)/2} \frac{1}{1 - 2^{(\beta-2)/2}} \le \frac{1}{2}.$$

Then, with help of Lemma A.2 (see appendix below) from (5.33) we conclude that

$$|\boldsymbol{B}_{r_i, X_0}| \le 1 + 2|\boldsymbol{B}_{\rho_0, X_0}| \quad \forall\, i \in \mathbb{N}, \tag{5.34}$$

what completes the proof of (5.29).

Now, let $\varepsilon > 0$ be arbitrarily chosen. According to (5.29) for every $X_0 \in \Sigma_\infty$ we may choose $0 < r = r(X_0) \le \varepsilon$ such that

$$r^{-\beta} \int\limits_{Q_r(X_0)} |\nabla B|^2 dx dt \ge \frac{1}{\varepsilon}.$$

Thus, by the Vitali covering lemma there exists a pairwise disjoint family $\{Q_{r_i}(X_i)\}$ ($r_i := r(X_i)$ such that $\{Q_{3r_i}(X_i)\}$ covers Σ_∞. Similarly to the above we conclude

$$\sum_{i=1}^{\infty} r_i^{\beta} \le c\varepsilon \|\nabla \boldsymbol{B}\|_2^2.$$

Thus, $d\mathcal{P}_\beta(\Sigma_\infty) = 0$, and the proof of the theorem is complete. ■

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A Appendix

For $X = (x,t), Y = (y,s) \in \mathbb{R}^{n+1}$ we define the parabolic metric

$$d_p(X,Y) = \max\{|x-y|, |s-t|^{\frac{1}{2}}\}, \quad X, Y \in \mathbb{R}^{n+1}.$$

Let $Q = \Omega \times (a,b)$, where $\Omega \subset \mathbb{R}^n$ is a bounded domain, and $-\infty < a < b < +\infty$. Then, for $0 < \gamma < 1$ we define the space of Hölder continuous functions on Q, $C^{\gamma, \frac{\gamma}{2}}(\bar{Q})$ by functions $f : \bar{Q} \to \mathbb{R}$ such that

$$[f]_{C^{\gamma, \frac{\gamma}{2}}} = \sup_{X, Y \in \bar{Q}, X \neq Y} \frac{|f(X) - f(Y)|}{d_p(X,Y)^\gamma} < +\infty.$$

The following parabolic version of the Poincare inequality has been proved in [22, Lemma B.3]

Lemma A.1 (Parabolic Poincaré-type inequality)**.** *Let* $Q_r = Q_r(X_0) \subset \mathbb{R}^{n+1}$ $(n \in \mathbb{N})$*. Let* $u \in L^p(Q_r)$ *be such that* $\nabla u \in L^p(Q_r)$ $(1 \le p < +\infty)$*. In addition suppose that there exists* $\boldsymbol{f} \in L^1(Q_r)^n$ *such that* $\partial_t u = \nabla \cdot \boldsymbol{f}$ *in sense of distributions, i. e.*

$$\int\limits_{Q_r} u\partial_t \varphi dxdt = \int\limits_{Q_r} \boldsymbol{f} \cdot \nabla\varphi dxdt \quad \forall\, \varphi \in C^\infty_{\rm c}(Q_r). \tag{A.1}$$

Then

$$⨍_{Q_r} |u - u_{Q_r}|^p dxdt \le cr^p ⨍_{Q_r} |\nabla u|^p dxdt + cr^p \bigg(⨍_{Q_r} |\boldsymbol{f}| dxdt \bigg)^p, \tag{A.2}$$

where $c = \text{const} > 0$*, depending on* n *and* p *only, but not on* r, u *or* $\boldsymbol{f}$*.*

The following elementary algebraic lemma has been used in the proof of Theorem 5.1.

Lemma A.2. *Let* $\{M_i\}$ *and* $\{\lambda_i\}$ *be sequences of positive numbers such that* $\sum \lambda_i \le \frac{1}{2}$*, and*

$$|M_{j+1} - M_j| \le (1 + M_j)\lambda_j \quad \forall\, j \in \mathbb{N}. \tag{A.3}$$

Then,

$$M_i \le 1 + 2M_1 \quad \forall\, i \in \mathbb{N}. \tag{A.4}$$

Proof We prove the statement of this lemma by induction. Cleary, for $i = 1$ the assertion is trivially fulfilled. Assume, (A.4) holds for $j = 1, \ldots, i$. Then, with help of of triangle inequality and (A.3) for $j = 1, \ldots, i$ we get

$$\begin{aligned} M_{i+1} &\le M_1 + |M_{i+1} - M_1| \le M_1 + \sum_{j=1}^{i} |M_{j+1} - M_j| \\ &\le M_1 + \sum_{j=1}^{i} (1 + M_j)\lambda_j \le M_1 + (2 + 2M_1) \sum_{j=1}^{i} \lambda_j \le 1 + 2M_1. \end{aligned}$$

Whence, the claim is proved. ∎

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